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# GENERALIZATIONS OF CONWAY'S TOPOGRAPH ARISING FROM ARITHMETIC COXETER GROUPS

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**Abstract**

Generalizations of Conway's Topograph arising from Arithmetic Coxeter Groups

by

Suzana Milea

Conway's topograph can be used in the study of binary quadratic forms (BQFs) to replace tedious algebraic computations with straightforward geometric arguments. The crux of his method is the isomorphism between the arithmetic group $PGL_2(\mathbb{Z})$ and the Coxeter group $(3, \infty)$. We introduce the arithmetic groups called dilinear groups and construct generalizations of Conway's topograph called dilinear topographs. Then we use them to study variants of BQFs, called binary quadratic diforms (BQDs). The payoff can be seen in the last chapter in our investigation of minimum value bounds for diforms and pairs of BQFs.

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# Introduction

Conway introduces in [2] a simple and elegant combinatorial-geometric method of classifying all integral binary quadratic forms (BQFs), and answering some basic questions about them. The geometry of Conway's topograph reflects the fact that $PGL_2(\mathbb{Z})$ is isomorphic to the Coxeter group of type $(3, \infty)$.

Let $\sigma > 1$ be a square-free positive integer. The dilinear group $DL_2(\mathbb{Z}[\sqrt{\sigma}])$ is the group of invertible matrices with entries in $\mathbb{Z}[\sqrt{\sigma}]$, where one diagonal has entries in $\mathbb{Z}$ and the other diagonal has entries in $\mathbb{Z} \cdot \sqrt{\sigma}$. Johnson and Weiss show in [5] that when $\sigma = 2$ or $\sigma = 3$ the dilinear groups admit Coxeter group presentations.

Whenever there is an isomorphism from a Coxeter group to an arithmetic group, it is natural to look for arithmetic interpretations. The coincidence between the dilinear groups and the Coxeter groups $(4, \infty)$ and $(6, \infty)$ led to the creation of the "dilinear topographs". These geometric objects can be used to study binary quadratic "diforms" and easily bound the minima of BQFs.

Here is a brief outline of each of the chapters in this thesis. Chapter 1 is collection of results and definitions necessary for later chapters. The notions of incidence system, incidence geometry and Coxeter geometry are introduced in Section 1.3.

Chapter 2 introduces the dilinear groups and describes their action on divectors. The goal is to give the group isomorphism from $(2\sigma, \infty)$ to $DL_2(R_\sigma)$ when $\sigma = 2, 3$. In Chapter 3 we introduce the dilinear topograph as an incidence system and prove that it is an incidence geometry (all maximal flags are chambers). We then use the group isomorphism described in Chapter 2 to prove that this geometry is isomorphic to the Coxeter geometry.

Chapter 4 introduces binary quadratic diforms, and their connection to BQFs. Their topographs exhibit similar features to Conway's topograph. We give the arithmetic progression property, climbing principle and the local formulas for the discriminant (from any cell in the topograph). We finish the chapter with a discussion on how the values from two Conway's topographs interlace in the topograph of a diform.

Chapter 5 is focused on nondegenerate indefinite diforms. Their topograph contain both positive and negative values. As in Conway's topograph, the river is the set of segments separating positive values from negative ones. Analyzing the shape of the river helps us determine minimum-value bounds for diforms and for pairs of related BQFs.

# Chapter 1

# Preliminaries

## 1.1 Reflection groups

The results presented in this section may be found in Chapter 7 of Ratcliffe's book *Foundations of Hyperbolic Manifolds* [9].

Let $X$ denote the unit $n$-sphere $S^n$, the Euclidean $n$-space $E^n$ or the hyperbolic $n$-space $\mathcal{H}^n$. Let $P$ be an $n$-dimensional convex polyhedron in $X$ and let $F$ be a facet of $P$ (i.e. a face of dimension $n-1$). The *reflection* of $X$ in the facet $F$ of $P$ is the reflection of $X$ in the hyperplane spanned by $F$.

**Definition 1.** A subset $R$ of a metric space $X$ is a *fundamental region* for a group $\Gamma$ of isometries of $X$ if and only if

(1) the set $R$ is open in $X$;

(2) the members of $\{gR : g \in \Gamma\}$ are mutually disjoint; and

(3) $X = \cup\{g\overline{R} : g \in \Gamma\}$. Here $\overline{R}$ denotes the closure of $R$.

**Definition 2.** A subset $D$ of a metric space $X$ is a *fundamental domain* for a group $\Gamma$ of isometries of $X$ if and only if $D$ is a connected fundamental region for $\Gamma$.

**Definition 3.** A fundamental region $R$ for a group $\Gamma$ of isometries of a metric space $X$ is *locally finite* if and only if $\{g\overline{R} : g \in \Gamma\}$ is a locally finite family of subsets of $X$ (i.e. for each point $x$ of $X$, there is an open neighborhood $U$ of $x$ in $X$ such that $U$ meets only finitely many members of the family).

**Definition 4.** A *convex fundamental polyhedron* for a discrete group $\Gamma$ of isometries of $X$ is a convex polyhedron $P$ in $X$ whose interior is a locally finite fundamental domain for $\Gamma$.

**Definition 5.** A convex fundamental polyhedron $P$ for a discrete group $\Gamma$ of isometries of $X$ is *exact* if for each facet $F$ of $P$ there is an element $g$ of $\Gamma$ such that $F = P \cap g(P)$.

**Theorem 6.** *([9], p.252) If $F$ is a facet of an exact, convex, fundamental polyhedron $P$ for a discrete group $\Gamma$ of isometries of $X$, then there is a unique element $g_F \neq 1$ of $\Gamma$ such that $F = P \cap g_F(P)$, moreover $g_F^{-1}(F)$ is a facet of $P$.*

The group $\Gamma$ is defined to be a *discrete reflection group*, with respect to the polyhedron $P$, if and only if $g_F$ is the reflection of $X$ in the hyperplane spanned by $F$ for each facet $F$ of $P$.

**Theorem 7.** *([9], p.265) Let $\Gamma$ be a discrete reflection group with respect to the polyhedron $P$. Then all the dihedral angles of $P$ are submultiples of $\pi$; moreover, if $g_{F_1}$ and $g_{F_2}$ are the reflections in adjacent facets $F_1$ and $F_2$ of $P$, and $\theta(F_1, F_2) = \pi/k$, then $g_{F_1}g_{F_2}$ has order $k$ in $\Gamma$.*

**Theorem 8.** *([9], p.265) Let $P$ be a finite-sided, $n$-dimensional, convex polyhedron in $X$ of finite volume all of whose dihedral angles are submultiples of $\pi$. Then the group $\Gamma$ generated by the reflections of $X$ in the facets of $P$ is a discrete reflection group with respect to the polyhedron $P$.*

**Theorem 9.** *([9], p.273) Let $\Gamma$ be a discrete reflection group with respect to a polyhedron $P$ in $X$ with finitely many facets and finite volume. Let $\{F_i\}$ be the set of facets of $P$ and for each pair of indices $i, j$ such that $F_i$ and $F_j$ are adjacent, let $k_{ij} = \pi/\theta(F_i, F_j)$. Then*

$$\langle F_i \mid F_i^2 = 1, (F_iF_j)^{k_{ij}} = 1 \rangle$$

*is a group presentation for $\Gamma$ under the mapping $F_i \mapsto g_{F_i}$.*

*Here it is understood that $(F_iF_j)^{k_{ij}}$ is to be deleted if $k_{ij} = \infty$.*

## 1.2 Coxeter groups

We will mainly follow the classical reference *Reflection groups and Coxeter groups* by Humphreys [4] to reproduce definitions and essential properties of Coxeter groups. The definition of a Coxeter group was motivated by finite groups generated by reflections and 'most' finite reflection groups turn out to be 'Weyl groups' (thus the letter $W$ is used).

**Definition 10.** A group $W$ is a *Coxeter group* if there is a finite subset $S$ of $W$ such that $W$ has the presentation

$$\langle s \in S \mid (ss')^{m(s,s')} = 1 \rangle$$

where $m(s,s') \in \{2,3,4,...,\infty\}$ is the order of $ss'$, $s \neq s'$, and $m(s,s) = 1$. (When $m(s,s') = \infty$ there is no relation between $s$ and $s'$).

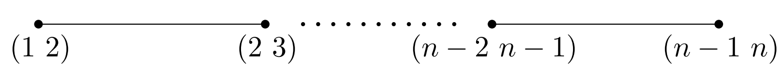
The pair $(W,S)$ is called a *Coxeter system.* The cardinality of $S$ is called the *rank* of $(W,S)$. Since the generators $s \in S$ have order 2 in $W$, each $w \neq 1$ in $W$ can be written in the form $w = s_1 s_2 \cdots s_r$ for some $s_i$ (not necessarily distinct) in $S$. If $r$ is as small as possible, call it the *length* of $w$, written $l(w)$.

A convenient way of describing a Coxeter system $(W,S)$ is through the construction of its *Coxeter graph.*

**Definition 11.** The *Coxeter graph* of the Coxeter system $(W,S)$ is an edge labelled graph $\Gamma_W$, with one node for each $s \in S$ and an edge from $s$ to $s'$ if $m(s,s') > 2$, labeled $m(s,s')$. (In practice, if $m(s,s') = 3$, the label is supressed).

**Example 12.** The symmetric group $\Sigma_n$ of permutations of $n$ letters is a Coxeter system when we let $S = \{(i\ i+1) : 1 \leq i < n\}$ be the set of elementary transpositions.

The Coxeter graph of $(\Sigma_n, S)$ is

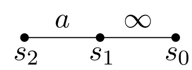



where the $i^{th}$ node corresponds to $(i\ i+1)$, $1 \leq i < n$.

**Example 13.** The group generated by $s_0, s_1, s_2$, subject to the relations

$$s_0^2 = s_1^2 = s_2^2 = (s_1 s_2)^a = (s_0 s_2)^2 = 1.$$

has the following graph




We will call this the Coxeter group **of type** $(a, \infty)$.

**Example 14.** Let $\Gamma$ be a discrete reflection group with respect to a finite-sided polyhedron $P$ of finite volume. Let $\{S_i\}$ be the set of facets of $P$, let $k_{ii} = 1$ for each $i$, and for each pair of indices $i, j$ such that $S_i$ and $S_j$ are adjacent, let $k_{ij} = \frac{\pi}{\theta(Si,Sj)}$, and let $k_{ij} = \infty$ otherwise. Let $s_i$ be the reflection corresponding to the facet $S_i$. Then Theorem 9 implies that $\Gamma$ is the Coxeter group with presentation

$$\langle s_i \mid (s_i s_j)^{k_{ij}} = 1 \rangle.$$

We can give a description of a Coxeter group as a motion group generated by mirror reflections through a hyperplane with respect to a bilinear form. We redefine a reflection to be merely a linear transformation which fixes a hyperplane pointwise and sends some nonzero vector to its negative.

**Definition 15.** Let $(W, S)$ be a Coxeter system. For a subset $T$ of $S$, let $W_T$ denote the subgroup of $W$ generated by $s \in T$ and $W^T$ denote the subgroup of $W$ generated by $s \in S \setminus T$. Any conjugate a subgroup of the form $W_T$ is called a *parabolic subgroup*.

**Theorem 16.** *([4], p.113) For each subset $T$ of $S$, the pair $(W_T, T)$ is a Coxeter system.*

We say a Coxeter system $(W, S)$ is *irreducible* if the Coxeter graph $\Gamma$ is connected.

**Theorem 17.** *([4], p.129) Let $(W, S)$ be any Coxeter system. If $\Gamma_1, \cdots, \Gamma_r$ are the connected components of the Coxeter graph $\Gamma$, let $S_1, \cdots, S_r$ be the corresponding subsets* *of $S$. Then $W$ is the direct product of the parabolic subgroups $W_{S_1}, \cdots, W_{S_r}$, and each Coxeter system $(W_{S_i}, S_i)$ is irreducible.*

## 1.3 Incidence geometry

The goal of this section is to define the notion of flag for a general Coxeter system $(W, S)$ (with finite $S$) and the incidence geometry of such flags. We are following Buekenhout and Cohen's book called *Diagram Geometry* [1].

**Definition 18.** Let $I$ be a set. A triple $\Gamma = (X, *, \tau)$ is called an *incidence system* over $I$ if

(1) $X$ is a set (its elements are also called elements of $\Gamma$ );

(2) $*$ is a symmetric and reflexive relation on $X$; it is called the *incidence relation* of $\Gamma$;

(3) $\tau$ is a map from $X$ to $I$, called the type map of $\Gamma$, such that distinct elements $x, y \in X$ with $x * y$ satisfy $\tau(x) \neq \tau(y)$; members of the pre-image $\tau^{-1}(i)$ are called elements of type $i$, or $i$-elements.

The set $I$ is called the *type* of $\Gamma$ and the cardinality of $I$ is called the *rank* of $\Gamma$. Its elements as well as its subsets are called *types*. If $A \subseteq X$, we say that $A$ is of type $\tau(A)$ and of rank $|\tau(A)|$, the cardinality of $\tau(A)$.

In an incidence system $\Gamma = (X, *, \tau)$ over $I$, the set $X$ is the disjoint union of the sets $X_i = \tau^{-1}(i)$, for $i \in I$. Thus, $(X, *)$ is a multipartite graph with partitioning $(X_i)_{i \in \tau(X)}$.

**Definition 19.** A *flag* of $\Gamma$ is a set of mutually incident elements of $\Gamma$. Flags of $\Gamma$ of type $I$ are called *chambers*.

*Remark* 20. A flag of $\Gamma$ has at most one element of each type.

*Remark* 21. By Zorn's lemma, every flag is contained in at least one maximal flag, that is, a flag not properly contained in any other flag. In an incidence system, chambers are maximal flags. In general, however, the converse does not hold.

**Definition 22.** Let $\Gamma$ be an incidence system over $I$. If every maximal flag of $\Gamma$ is a chamber, then $\Gamma$ is called a *geometry* over $I$.

**Definition 23.** Let $\Gamma = (X, *, \tau)$ be an incidence system over $I$ and $\Gamma' = (X', *', \tau')$ an incidence system over $I'$. A *weak homomorphism* $\alpha : \Gamma \to \Gamma'$ is a map $\alpha : X \to X'$ such that, for all $x, y \in X$,

(1) $x * y$ implies $\alpha(x) *' \alpha(y)$ (i.e. $\alpha$ preserves incidence);

(2) $\tau(x) = \tau(y)$ implies $\tau'(\alpha(x)) = \tau'(\alpha(y))$ (i.e. $\alpha$ sends elements of the same type in $I$ to elements of the same type in $I'$).

If, in addition, $I = I'$ and $\tau(x) = \tau'(\alpha(x))$ for all $x \in X$, then $\alpha$ is called a *homomorphism*. A bijective weak homomorphism $\alpha$ whose inverse $\alpha^{-1}$ is also a weak homomorphism is called a *correlation*. If $\alpha$ is a homomorphism and a correlation, then we call $\alpha$ an *isomorphism* (of incidence systems) and write $\Gamma \cong \Gamma'$.

**Definition 24.** Let $(G_i)_{i \in I}$ be a system of subgroups of the group $G$. The *coset incidence system* of $G$ over $(G_i)_{i \in I}$, denoted by $\Gamma(G, (G_i)_{i \in I})$ is the incidence system over $I$, whose elements of type $i$ are the cosets of $G_i$ in $G$ and in which the incidence relation is given by

$$aG_i \text{ and } bG_j \text{ are incident if and only if } aG_i \cap bG_j \neq \emptyset.$$

If $\Gamma(G,(G_i)_{i\in I})$ is a geometry it is called the *coset geometry*.

*Remark* 25. In Definition 24 we say "system" of subgroups rather than "set" of subgroups to prevent the confusion in case two subgroups $G_j$ and $G_k$ are the same for distinct $j,k \in I$. A coset of $G_j$ coincides with a coset of $G_k$ only if $G_j = G_k$. So, if the subgroups of the system are chosen to be mutually distinct, the union of $G/G_i$ over all $i \in I$ is disjoint. Furthermore, an instance $G_j = G_k$ for distinct $j$ and $k$ does not provide an interesting geometry.

*Remark* 26. $G$ acts on the coset incidence system $\Gamma(G,(G_i)_{i\in I})$ by left multiplication. The notation $G_i$ has been chosen so as to resemble the notation for the stabilizer in $G$ of an element of type $i$. Indeed, $G_i$ is the stabilizer of the element $G_i$ of type $i$ and $\{G_i | i \in I\}$ is a chamber of $\Gamma(G,(G_i)_{i\in I})$.

**Definition 27.** Given subgroups $G_i$ $(i \in I)$ of $G$, and $J \subseteq I$, we write $G_J$ to denote $\bigcap_{j\in J} G_j$. We call this subgroup the *standard parabolic subgroup* of $G$ of type $J$ (so $G_{\{j\}} = G_j$ for each $j \in J$).

The following result gives three equivalent conditions necessary for a coset incidence system to be a geometry.

**Theorem 28.** *([1], p.33) Let $\Gamma$ be the coset incidence system of $G$ over $(G_i)_{i\in I}$. If $I$ is finite, then the following statements are equivalent.*

*(i) $G$ is flag transitive on $\Gamma$.*

*(ii) For each subset $J$ of $I$ of size three, the group $G$ is transitive on the set of flags of type $J$, and for each $i \in I$ the subgroup $G_i$ is flag transitive on $\Gamma(G_i, (G_{\{i,j\}})_{j \in I \setminus \{i\}})$.*

*(iii) For each $J \subseteq I$ and each $i \in I \setminus J$, we have $G_J G_i = \bigcap_{j \in J}(G_j G_i)$.*

*If one (whence all) of these properties hold, then $\Gamma$ is a geometry.*

Let $I$ be a finite set and $(W, S = \{s_i\}_{i \in I})$ be a Coxeter system. Let $W^i$ denote the parabolic subgroup of $W$ generated by $s \in S \setminus \{s_i\}$. Let $\Gamma_W$ be the coset incidence system of $W$ over $\{W^i\}_{i \in I}$. In this case condition (iii) of the above theorem is exactly the statement of the following result. Thus $\Gamma_W$ is a geometry (called *Coxeter geometry*).

**Proposition 29.** *([10], Prop.2.2.12) Let $(W, S)$ be a Coxeter system of finite rank. For any proper non-empty subset $T \subseteq S$, and for any $s \in S \setminus T$,*

$$W^T W^s = \bigcap_{t \in T} W^t W^s.$$

## 1.4 Conway's topograph

Parts of this section have already appeared in our article in PNAS [8]. The purpose of this section is to describe the combinatorial-geometric method for analyzing integer-valued binary quadratic forms introduced in Conway's book *The Sensual (Quadratic) Form* [2].

Recall that binary quadratic forms (BQFs) are functions $Q : \mathbb{Z}^2 \to \mathbb{Z}$ of the form $Q(x, y) = ax^2 + bxy + cy^2$; with $a, b, c \in \mathbb{Z}$. Since $Q(kx, ky) = k^2 Q(x, y)$, to understand the values of $Q$ at all vectors in $\mathbb{Z}^2$ it will suffice to explore its values at *primitive* vectors, i.e. vectors $\vec{v} = (a, b) \in \mathbb{Z}^2$ with the property that $a$ and $b$ are coprime integers. Also, since $Q(-\vec{v}) = Q(\vec{v})$ it will be convenient to think of $\vec{v}$ and $-\vec{v}$ as the same vector. Such a vector will be written $\pm\vec{v}$ and called a *lax* vector.

**Definition 30.** A *lax basis* is an unordered pair $\{\pm\vec{v}, \pm\vec{w}\}$ of primitive lax vectors which form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

**Definition 31.** A *lax superbasis* is an unordered triple $\{\pm\vec{u}, \pm\vec{v}, \pm\vec{w}\}$, any two of which form a lax basis.

**Definition 32.** The *topograph* is the incidence system of type $I = \{0, 1, 2\}$, consisting of: faces (elements of type 2) are primitive lax vectors, edges (elements of type 1) are lax bases, and points (elements of type 0) are lax superbases. Incidence among points, edges, and faces is defined by containment (symmetrically).

A *maximal flag* in this context refers to a point contained in an edge contained in a face. Conway shows in his book ([2]) how every partial flag can be completed to a chamber. He describes how primitive lax vectors can be completed to lax bases, then to lax superbases. Since every maximal flag is a chamber, Conway's topograph is a geometry over $I$. The geometry is displayed in Figure 1.1; the points and edges form a ternary regular tree, and the faces are $\infty$-gons.

The geometry of Figure 1.1 also arises as the coset geometry of the *Coxeter group* $W$ of type $(3, \infty)$ generated by the set $S = \{s_i\}_{i \in I}$ where $I = \{0, 1, 2\}$. For a subset $J$ of $I$, the flags of type $J$ are cosets $W/W^J$. The maximal flags are the flags of type $I$, i.e., the cosets $W/W^I = W/W_\emptyset = W/\{Id\} = W$. Since the action of $W$ on $W$

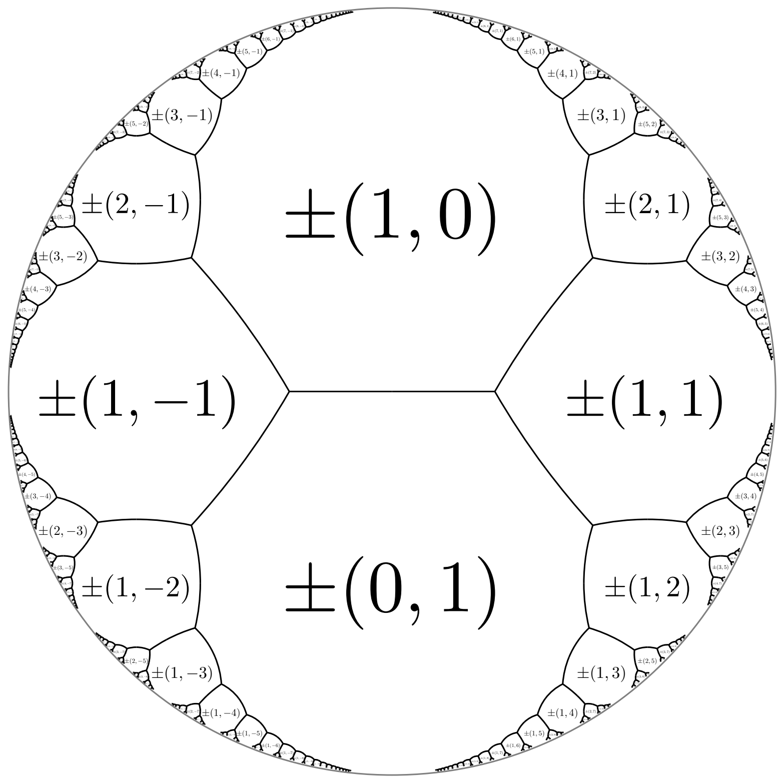



Figure 1.1: Conway's geometry of primitive lax vectors, lax bases, and lax superbases.

by left-multiplication is simply-transitive, the Coxeter group $W$ acts simply-transitively on the maximal flags.

The geometric coincidence reflects the fact that $PGL_2(\mathbb{Z})$ is isomorphic to the Coxeter group $W$ of type $(3, \infty)$, a classical result known to Poincaré and Klein. This raised the natural question: given a coincidence between an arithmetic group and a Coxeter group, is there an arithmetic interpretation of the flags in the Coxeter group?

## 1.5 Topographs of binary quadratic forms

Parts of this section have already appeared in our article in PNAS [8]. A detailed description of topographs of BQFs can be found in Chapter III of the book *An Illustrated Theory of Numbers* [11].

Let $Q$ be a BQF. We can obtain Conway's *topograph* of $Q$ by labeling the faces of the topograph: the face corresponding to the primitive lax vector $\pm\vec{v}$ is labeled by the value $Q(\pm\vec{v})$. Figures 1.2 and 1.3 display examples.

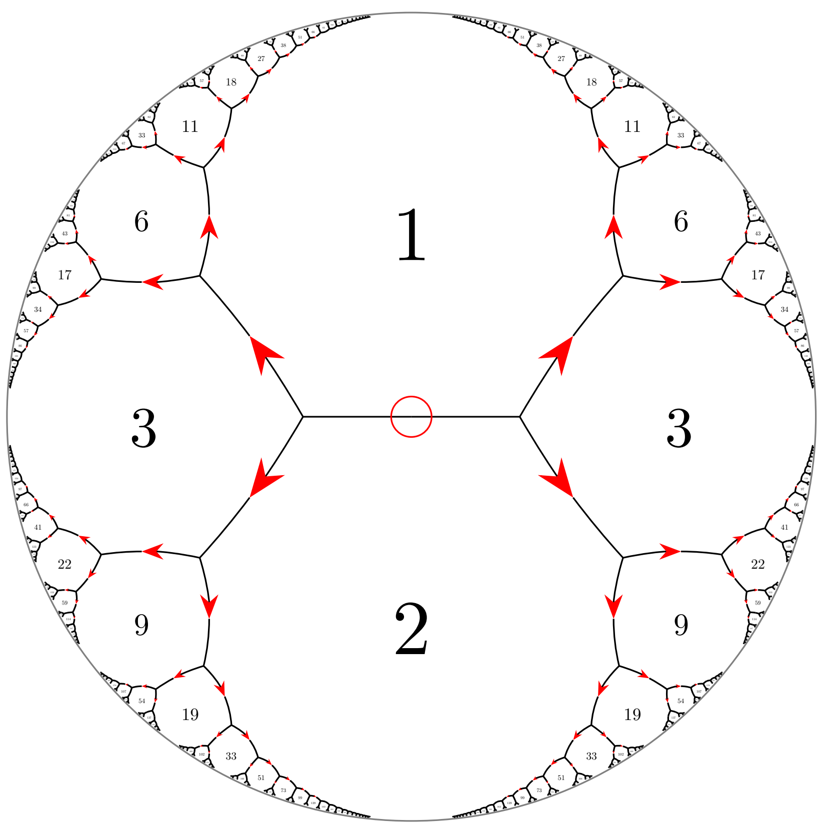
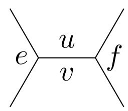



Figure 1.2: The topograph of $Q(x, y) = x^2 + 2y^2$, with arrows exhibiting the climbing principle. The well (source of the flow) is the cell at the center of the figure.

If $u, v, e, f$ appear on the topograph of $Q$, in a local arrangement we call a *cell*, then Conway observes that the integers $e, u+v, f$ form an arithmetic progression.

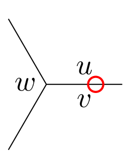



$$f - (u+v) = (u+v) - e.$$

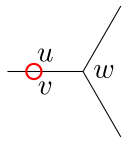
The discriminant of $Q$ can be seen locally in the topograph, at every cell, by the formula

$$\Delta = u^2 + v^2 + e^2 - 2uv - 2ve - 2eu = (u-v)^2 - ef.$$

A consequence of the arithmetic progression property is Conway's climbing principle; if all values in a cell are positive, place arrows along the edges in the directions of increasing arithmetic progressions. Then every arrow propagates into two arrows; the resulting *flow* along the edges can have a source, but never a sink. This implies the existence and uniqueness of a *well* for positive-definite forms: a triad or cell which is the source for the flow. The well gives the unique Gauss-reduced form $Q_{\mathrm{Gr}}$ in the $SL_2(\mathbb{Z})$-equivalence class of $Q$. More precisely, every well contains a triple $u \leq v \leq w$ of positive integers satisfying $u+v \geq w$, with strict inequality at triad-wells and equality at cell-wells (see Figure 1.2). Depending on the orientation of $u, v, w$ at the well, the Gauss-reduced form is given below; in the ambiguously-oriented case with $u = v$, $Q_{\mathrm{Gr}}(x,y) = ux^2 + (u+v-w)xy + vy^2$. If $u + v = w$, both orientations occur in a cell-well, and $Q_{\mathrm{Gr}}(x,y) = ux^2 + vy^2$.




$$\longmapsto Q_{\mathrm{Gr}}(x,y) = ux^2 + (u+v-w)xy + vy^2$$




$$\longmapsto Q_{\mathrm{Gr}}(x,y) = ux^2 - (u+v-w)xy + vy^2$$

When $Q$ is a nondegenerate indefinite form, Conway defines the *river* of $Q$ to be the set of edges which separate a positive value from a negative value in the topograph of $Q$. Since all values on the topograph of $Q$ must be positive or negative, the river cannot branch or terminate. The climbing principle implies uniqueness of the river. Thus the river is a set of edges comprising a single endless line. Bounding the values adjacent to the river implies periodicity of values adjacent to the river, and thus the infinitude of solutions to Pell's equation. This is described in detail in [2]. *Riverbends* – cells with a river as drawn below – correspond to Gauss's reduced forms in the equivalence class of $Q$.


The existence of riverbends gives a classical bound, by an argument we learned from Gordan Savin.

**Theorem 33.** *If $Q$ is a nondegenerate indefinite BQF, then the minimum nonzero value $\mu_Q$ of $Q$ satisfies $|\mu_Q| \leq \sqrt{\Delta/5}$.*

*Proof.* At a riverbend, one finds $\Delta = (u-v)^2 - ef = u^2 + v^2 - uv - vu - ef$, the sum of five *positive* integers. It follows that one of $u^2, v^2, -uv, -vu, -ef$ must be bounded by $\Delta/5$. Among $|u|, |v|, |e|, |f|$, one must be bounded by $\sqrt{\Delta/5}$. □

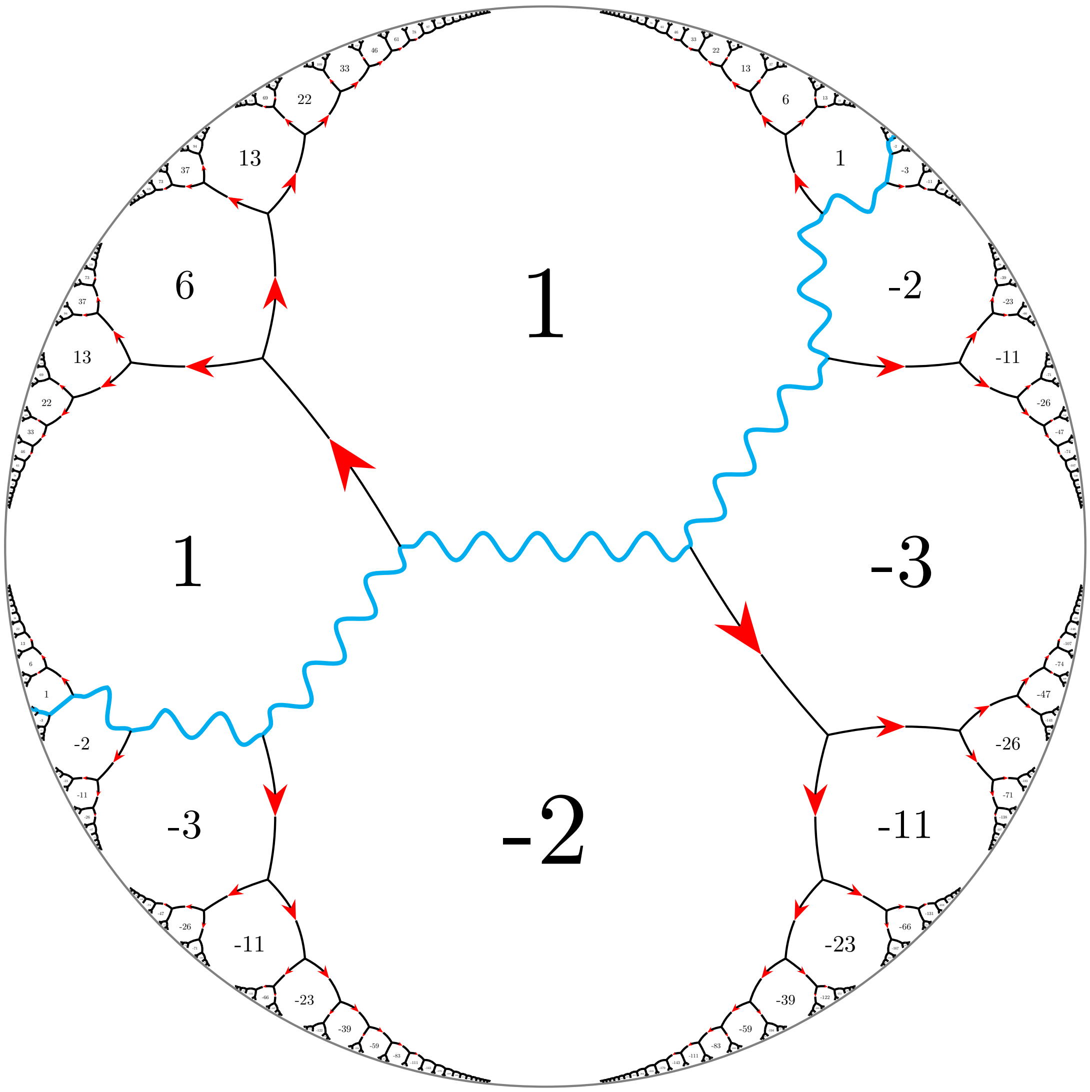



Figure 1.3: The topograph of $Q(x, y) = x^2 - 3y^2$, exhibiting a periodic river. Solutions to Pell's equation $x^2 - 3y^2 = 1$ are found along the riverbank.

# Chapter 2

# Dilinear groups

The goal of this chapter is to prove there is an isomorphism from the Coxeter group $(2\sigma, \infty)$ to the arithmetic group $PDL_2(R_\sigma) = DL_2(R_\sigma)/\{\pm\mathbf{1}\}$ when $\sigma = 2, 3$. This isomorphism was what made us look for applications of Coxeter groups to arithmetic. Here we introduce the dilinear group $DL_2(R_\sigma)$, describe its generators and prove the desired result.

## 2.1 Dilinear algebra

**Definition 34.** Let $\sigma > 1$ be a square-free positive integer, and let $R_\sigma = \mathbb{Z}[\sqrt{\sigma}]$ be the quadratic ring of discriminant $4\sigma$. We define the *dilinear group* $DL_2(R_\sigma)$ to be the group of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R_\sigma) \text{ such that } (a, d \in \mathbb{Z}\cdot\sqrt{\sigma} \text{ and } b, c \in \mathbb{Z}) \text{ or } (a, d \in \mathbb{Z} \text{ and } b, c \in \mathbb{Z}\cdot\sqrt{\sigma}).$$

Moreover, we define $PDL_2(R_\sigma) = DL_2(R_\sigma)/\{\pm\mathbf{1}\}$.

**Notation 35.** *Let $DL_2^-(R_\sigma)$ be the subset of $DL_2(R_\sigma)$ consisting of matrices with $a, d \in \mathbb{Z} \cdot \sqrt{\sigma}$ and $b, c \in \mathbb{Z}$ and let $DL_2^+(R_\sigma)$ be the subset of $DL_2(R_\sigma)$ consisting of matrices with $a, d \in \mathbb{Z}$ and $b, c \in \mathbb{Z} \cdot \sqrt{\sigma}$.*

*Remark* 36. $DL_2^+(R_\sigma)$ is an index-two (hence normal) subgroup of $DL_2(R_\sigma)$ and $DL_2^-(R_\sigma)$ is its nontrivial coset. Morevoer, $DL_2^+(R_\sigma)$ is $GL_2(\mathbb{Q}(\sqrt{\sigma}))$-conjugate to a congruence subgroup of $GL_2(\mathbb{Z})$: if $g = \operatorname{diag}(1, \sqrt{\sigma})$, then

$$gDL_2^+(R_\sigma)g^{-1} = \Gamma_0(\sigma) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{Z}) : \gamma \in \sigma\mathbb{Z} \right\}.$$

**Lemma 37.** *Suppose that $u, v \in \mathbb{Z}$. Then*

$$GCD(u, v\sqrt{\sigma}) = 1 \textit{ in } R_\sigma \textit{ if and only if } \operatorname{GCD}(u, \sigma v) = 1 \textit{ in } \mathbb{Z}.$$

*Proof.* Recall that $GCD(u, v\sqrt{\sigma}) = 1$ in $R_\sigma$ means that the pair $\{u, v\sqrt{\sigma}\}$ generates the unit ideal in $R_\sigma$. That is, there exist $x + y\sqrt{\sigma}, s + r\sqrt{\sigma} \in R_\sigma$ such that

$$(x + y\sqrt{\sigma})u + (s + r\sqrt{\sigma})v\sqrt{\sigma} = 1,$$

which we can rewrite as

$$(xu + r\sigma v) + (yu + sv)\sqrt{\sigma} = 1.$$

This is equivalent to $xu + rv\sigma = 1$ and $yu + sv = 0$. In other words, there exist $x, r \in \mathbb{Z}$ such that $xu + r(\sigma v) = 1$. That is, $\operatorname{GCD}(u, \sigma v) = 1$ in $\mathbb{Z}$. □

**Definition 38.** A *divector* over $R_\sigma$ will mean a vector in $R_\sigma^2$ of *red* or *blue* type. Red divectors are those of the form $(u, v\sqrt{\sigma})$ for some $u, v \in \mathbb{Z}$. Blue divectors are those of the form $(u\sqrt{\sigma}, v)$ for some $u, v \in \mathbb{Z}$. Let $R_\sigma^{\text{di}}$ denote the set of divectors and let $R_\sigma^{\text{red}}$ and $R_\sigma^{\text{blue}}$ denote, respectively, its subsets of red and blue divectors.

A red divector $(u, v\sqrt{\sigma})$ is called *primitive* if $\text{GCD}(u, \sigma v) = 1$. A blue divector $(u\sqrt{\sigma}, v)$ is called *primitive* if $\text{GCD}(u\sigma, v) = 1$.

**Theorem 39.** *The dilinear group $DL_2(R_\sigma)$ acts (by matrix multiplication) transitively on the set of primitive divectors, and its subgroup $DL_2^+(R_\sigma)$ acts transitively on the set of primitive red (or blue) divectors.*

*Proof.* To see that $DL_2(R_\sigma)$ acts on the set of primitive divectors, note that a matrix in $DL_2(R_\sigma)$ is invertible and its determinant is an integer. Hence a matrix in $DL_2(R_\sigma)$ has determinant equal to $\pm 1$ and thus it sends primitive divectors to primitive divectors.

It remains to prove that the action of $DL_2(R_\sigma)$ on primitive divectors is transitive. Since the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in DL_2(R_\sigma)$ swaps primitive red and blue divectors, it suffices to show that $DL_2^+(R_\sigma)$ acts transitively on the set of primitive red vectors.

Let $(u, v\sqrt{\sigma})$ be a primitive red divector. Since $\text{GCD}(u, \sigma v) = 1$, there exist $s, t \in \mathbb{Z}$ such that $su - tv\sigma = 1$ and

$$\begin{pmatrix} u & t\sqrt{\sigma} \\ v\sqrt{\sigma} & s \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v\sqrt{\sigma} \end{pmatrix}.$$

In other words, for any primitive red divector $(u, v\sqrt{\sigma})$ there exists a matrix $M \in DL_2^+(R_\sigma)$ such that $M \cdot \binom{1}{0} = \binom{u}{v\sqrt{\sigma}}$.

□

*Remark* 40. The rows and columns of a matrix in $DL_2(R_\sigma)$ are primitive divectors.

## 2.2 Generators for $PDL_2(R_2)$ and $PDL_2(R_3)$

When $\sigma = 2$ or $\sigma = 3$, Johnson and Weiss [5, §4] present $PDL_2(R_\sigma)$ by generators and relations, giving an isomorphism between $PDL_2(R_\sigma)$ and the Coxeter group $(2\sigma, \infty)$. In this section we give an algebraic proof of their claim about the set of generators for the dilinear groups.

**Theorem 41.** *If $\sigma = 2$ or $\sigma = 3$, then $DL_2(R_\sigma)$ is generated by the triple of matrices,*

$$r_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} -1 & 0 \\ \sqrt{\sigma} & 1 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We will use the modified division algorithm for integers (which gives the remainder of least absolute value) to help us determine the quotient and remainder when dividing $a$ by $b\sqrt{\sigma}$ or $a\sqrt{\sigma}$ by $b$, where $a, b \in \mathbb{Z}$.

**Lemma 42** (Division with remainder in $\mathbb{Z} \sqcup \mathbb{Z}\sqrt{\sigma}$)**.** *Let $\sigma = 2, 3$.*

*(a) For all $a \in \mathbb{Z}$, $b\sqrt{\sigma} \in \mathbb{Z}\sqrt{\sigma}$, $b \neq 0$, there exist $q\sqrt{\sigma} \in \mathbb{Z}\sqrt{\sigma}, r \in \mathbb{Z}$ such that*

$$a = q\sqrt{\sigma} \cdot b\sqrt{\sigma} + r \text{ and } |r| < |b\sqrt{\sigma}|.$$

*(b) For all $a\sqrt{\sigma} \in \mathbb{Z}\sqrt{\sigma}$, $b \in \mathbb{Z}$, $b \neq 0$ there exist $q\sqrt{\sigma}, r\sqrt{\sigma} \in \mathbb{Z}\sqrt{\sigma}$ such that*

$$a\sqrt{\sigma} = q\sqrt{\sigma} \cdot b + r\sqrt{\sigma} \text{ and } |r\sqrt{\sigma}| < |b|.$$

*Proof.* (a) By the modified division algorithm, given integers $a$ and $b\sigma$ with $b \neq 0$ there exist $q, r \in \mathbb{Z}$ such that $a = q \cdot b\sigma + r$ and $|r| \leq |\frac{b\sigma}{2}|$. Then we obtain

$a = q\sqrt{\sigma} \cdot b\sqrt{\sigma} + r$ by simply rewriting $\sigma$. Moreover, since $\frac{\sigma}{2} < \sqrt{\sigma}$ for $\sigma = 2, 3$, we get that $|r| \leq |\frac{b\sigma}{2}| < |b\sqrt{\sigma}|$.

(b) By the modified division algorithm, given $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, $b \neq 0$, there exist $q, r \in \mathbb{Z}$ such that $a = q \cdot b + r$ and $|r| \leq |\frac{b}{2}|$. Then we get $a\sqrt{\sigma} = q\sqrt{\sigma} \cdot b + r\sqrt{\sigma}$ by simply multiplying both sides by $\sqrt{\sigma}$. Moreover, since $\frac{\sqrt{\sigma}}{2} < 1$ for $\sigma = 2, 3$, we get that $|r\sqrt{\sigma}| \leq |\frac{b\sqrt{\sigma}}{2}| < |b|$.

□

**Definition 43.** For a red divector $\alpha_r = (u, v\sqrt{\sigma})$ and a blue divector $\alpha_b = (u\sqrt{\sigma}, v)$ we define

$$\text{size}(\alpha_r) = \max(|u|, |v\sqrt{\sigma}|) \text{ and } \text{size}(\alpha_b) = \max(|u\sqrt{\sigma}|, |v|).$$

We can now give the proof of the main result of this section.

*Proof.* (of Theorem 41)

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in DL_2(R_\sigma)$. Then either $\gamma \in DL_2^+(R_\sigma)$ or $\gamma \in DL_2^-(R_\sigma)$.

We start by writing down the effect of multiplying $\gamma$ from the left by some elements of the subgroup generated by $r_0, r_1$ and $r_2$:

- $r_0$ changes sign in the first row,
- $r_2$ switches rows,
- $r_1 r_0 = \begin{pmatrix} 1 & 0 \\ -\sqrt{\sigma} & 1 \end{pmatrix}$ subtracts $\sqrt{\sigma}$ times the first row from the second row, and

- $r_2 r_1 r_0 r_2 = \begin{pmatrix} 1 & -\sqrt{\sigma} \\ 0 & 1 \end{pmatrix}$ subtracts $\sqrt{\sigma}$ times the second row from the first row.

Since $r_2$ transforms a matrix in $DL_2^-(R_\sigma)$ into a matrix in $DL_2^+(R_\sigma)$, it is enough to consider only matrices in $DL_2^+(R_\sigma)$. Let $\gamma = \begin{pmatrix} u & x\sqrt{\sigma} \\ v\sqrt{\sigma} & y \end{pmatrix} \in DL_2^+(R_\sigma)$.

Let $\alpha = \binom{u}{v\sqrt{\sigma}}$ denote the first column of $\gamma$. We can assume $u$ and $v$ are non-negative integers since we can use $r_0$ and $r_2$ to change sign. Moreover, since $\gamma \in GL_2(R_\sigma)$, the entries in a column of $\gamma$ cannot be both zero.

If either of the entries of the primitive divector $\alpha$ is zero then we must have $\alpha = \binom{1}{0}$. This means that $\gamma$ is a matrix of the form $\begin{pmatrix} 1 & x\sqrt{\sigma} \\ 0 & y \end{pmatrix}$. Since its determinant must be in $\mathbb{Z} \cap \mathbb{Z}[\sqrt{\sigma}]^\times = \{\pm 1\}$ we have $y = \pm 1$. Thinking about the effect of $r_0, r_2$ and powers of $r_1 r_0$ on a matrix we can easily write

$$(r_1 r_0)^x r_2 \begin{pmatrix} 1 & x\sqrt{\sigma} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } (r_1 r_0)^x r_0 r_2 \begin{pmatrix} 1 & x\sqrt{\sigma} \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is, $\begin{pmatrix} 1 & x\sqrt{\sigma} \\ 0 & 1 \end{pmatrix} = r_2 (r_1 r_0)^{-x}$ and $\begin{pmatrix} 1 & x\sqrt{\sigma} \\ 0 & -1 \end{pmatrix} = r_2 r_0 (r_1 r_0)^{-x}$.

Now we assume that both entries of $\alpha = \binom{u}{v\sqrt{\sigma}}$ are nonzero. We want to show that we can multiply $\alpha$ on the left by enough copies of $r_0, r_1$ and $r_2$ to obtain a vector of smaller size. If we can reduce the size then we can eventually obtain the divector $\binom{1}{0}$ of smallest possible size (size one). This means that eventually multiplication on the left by enough copies of $r_0, r_1$ and $r_2$ gives a matrix of the form $\begin{pmatrix} 1 & x\sqrt{\sigma} \\ 0 & y \end{pmatrix}$. We showed in the above paragraph how this type of matrix can be written as a product of the matrices $r_0, r_1$ and $r_2$.

If $v\sqrt{\sigma} > u$, by Lemma 42 there exist $q\sqrt{\sigma}, r\sqrt{\sigma} \in \mathbb{Z}\sqrt{\sigma}$ such that

$$v\sqrt{\sigma} = q\sqrt{\sigma} \cdot u + r\sqrt{\sigma} \text{ and } |r\sqrt{\sigma}| < |u|.$$

Then

$$(r_1 r_2)^q \alpha = \begin{pmatrix} u \\ v\sqrt{\sigma} - q\sqrt{\sigma} \cdot u \end{pmatrix} = \begin{pmatrix} u \\ r\sqrt{\sigma} \end{pmatrix} = \beta \text{ and } \operatorname{size}(\beta) < \operatorname{size}(\alpha).$$

If $v\sqrt{\sigma} < u$, by Lemma 42 there exist $q\sqrt{\sigma} \in \mathbb{Z}\sqrt{\sigma}, r \in \mathbb{Z}$ such that

$$u = q\sqrt{\sigma} \cdot v\sqrt{\sigma} + r \text{ and } |r| < |v\sqrt{\sigma}|.$$

Then

$$(r_0 r_1 r_2 r_0)^q \alpha = \begin{pmatrix} u - q\sqrt{\sigma} \cdot v\sqrt{\sigma} \\ v\sqrt{\sigma} \end{pmatrix} = \begin{pmatrix} r \\ v\sqrt{\sigma} \end{pmatrix} = \beta \text{ and } \operatorname{size}(\beta) < \operatorname{size}(\alpha).$$

□

## 2.3 Dilinear groups and Coxeter groups

Replacing each matrix $r_i$ in Theorem 41 by the equivalence class $\rho_i$ of matrices $\pm r_i$ we obtain the following result.

**Theorem 44.** *If $\sigma = 2$ or $\sigma = 3$, then $PDL_2(R_\sigma)$ is generated by*

$$\rho_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \rho_1 = \begin{bmatrix} -1 & 0 \\ \sqrt{\sigma} & 1 \end{bmatrix}, \rho_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Recall that the Coxeter group $(2\sigma, \infty)$ is the group generated by $s_0, s_1, s_2$, subject to the relations

$$s_0^2 = s_1^2 = s_2^2 = 1, (s_1 s_2)^{2\sigma} = 1, (s_0 s_2)^2 = 1.$$

Define $\phi : (2\sigma, \infty) \to PDL_2(R_\sigma)$ by $s_0 \mapsto \rho_0, s_1 \mapsto \rho_1$ and $s_2 \mapsto \rho_2$. It is easy to check that the generators of $PDL_2(R_\sigma)$ satisfy the Coxeter relations $\rho_0^2 = \rho_1^2 = \rho_2^2 = \mathbf{1}$, $(\rho_1\rho_2)^{2\sigma} = \mathbf{1}$, $(\rho_0\rho_2)^2 = \mathbf{1}$. Thus $\phi$ is a surjective homomorphism.

The goal of this section is to show that $\phi$ is an isomorphism if $\sigma = 2$ or $\sigma = 3$. To accomplish this, we will show that $\rho_0, \rho_1$ and $\rho_2$ act as reflections in the sides of a hyperbolic triangle.

Let $\mathcal{H} = \{x+iy \mid y > 0\}$ denote the Poincaré upper half plane. Recall that lines in $\mathcal{H}$ are Euclidean semicircles with centres on $x$-axis or Euclidean lines perpendicular to the $x$-axis.

A *hyperbolic reflection* is either a Euclidean reflection in a vertical line or an inversion centered at some point on the $x$-axis (when the hyperbolic line is represented by a semicircle).

The *inversion* with respect to the circle $C$ with center $(c, 0)$ and radius $r$ is the mapping $f : C \cup \{\infty\} \to C \cup \{\infty\}$ interchanging the points $c$ and $\infty$, and such that for each point $z \in \mathbb{C} \setminus \{c\}$, $f(z) = w$ lies in the line determined by $z$ and $c$, in such a way that

$$|z - c| \cdot |w - c| = r^2.$$

The circle $C$ can be recovered as the fixed point set for $f$. The general formula for the inversion $f$ across a circle with radius $r$ and center $(c, 0)$ is given by:

$$f(z) = r^2(\overline{z - c})^{-1} + c.$$

Let $\text{Isom}(\mathcal{H})$ denote the group of isometries of the hyperbolic plane. Let $PS^*L_2(\mathbb{R}) = S^*L_2(\mathbb{R})/\{\pm\mathbf{1}\}$ where $S^*L_2(\mathbb{R})$ is the group of real matrices of determinant $\pm 1$. For $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL_2(\mathbb{R})$ define

$$\gamma(z) = \begin{cases} (az + b)(cz + d)^{-1} \text{ if } \det(\gamma) > 0 \\ (a\bar{z} + b)(c\bar{z} + d)^{-1} \text{ if } \det(\gamma) < 0. \end{cases}$$

This defines an isomorphism from $PS^*L_2(\mathbb{R})$ to $\text{Isom}(\mathcal{H})$ (Theorem 1.3.1, [6]). Since $PDL_2(R_\sigma)$ is a subgroup of $PS^*L_2(\mathbb{R})$, any $\gamma \in PDL_2(R_\sigma)$ is uniquely determined by the corresponding isometry of the hyperbolic plane.

The following equations show that $\rho_0$ acts as the reflection in the $y$-axis, $\rho_1$ acts as the inversion across the circle of radius $\frac{1}{\sqrt{\sigma}}$ centered at $\left(-\frac{1}{\sqrt{\sigma}}, 0\right)$ and $\rho_2$ acts as the inversion across the unit circle centered at origin.

$$\rho_0(z) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z = -\bar{z},$$

$$\rho_1(z) = \begin{bmatrix} -1 & 0 \\ \sqrt{\sigma} & 1 \end{bmatrix} z = (-\bar{z})(\sqrt{\sigma}\bar{z} + 1)^{-1} = \left(\frac{1}{\sqrt{\sigma}}\right)^2 \left(\overline{z + \frac{1}{\sqrt{\sigma}}}\right)^{-1} - \frac{1}{\sqrt{\sigma}},$$

$$\rho_2(z) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z = (\bar{z})^{-1}.$$

More precisely, $\rho_0, \rho_1$ and $\rho_2$ act as reflections in the sides of a $(\frac{90}{\sigma})^\circ - 90^\circ - 0^\circ$ hyperbolic triangle. See the triangle $OAB$ in Figure 2.1 and Figure 2.2 in the case $\sigma = 2$ and $\sigma = 3$, respectively.

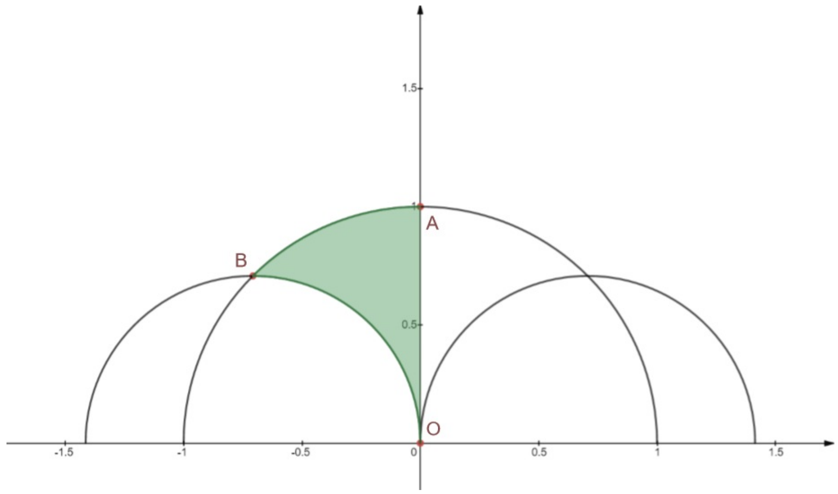



Figure 2.1: $45^\circ - 90^\circ - 0^\circ$ triangle

Let $T$ denote the hyperbolic triangle $OAB$ and let $S_0 = OA$, $S_1 = OB$ and $S_2 = AB$ denote its sides. Then $\rho_i$ corresponds to the reflection in the side $S_i$ for $i \in \{0, 1, 2\}$. The angles of $T$ are $\theta(S_1, S_2) = \pi/2\sigma$, $\theta(S_0, S_2) = \pi$ and $\theta(S_0, S_1) = 0$. Since they are all submultiples of $\pi$, Theorem 8 implies that the group generated by the reflections in the sides of the triangle $T$ (i.e $PDL_2(R_\sigma)$) is a discrete reflection group with respect to $T$.

Moreover, for each pair of indices $i, j$ such that $S_i$ and $S_j$ are adjacent, let $k_{ij} = \pi/\theta(S_i, S_j)$. That is, $k_{12} = \pi/\theta(S_1, S_2) = 2\sigma$, $k_{02} = \pi/\theta(S_0, S_2) = 2$ and $k_{01} =$

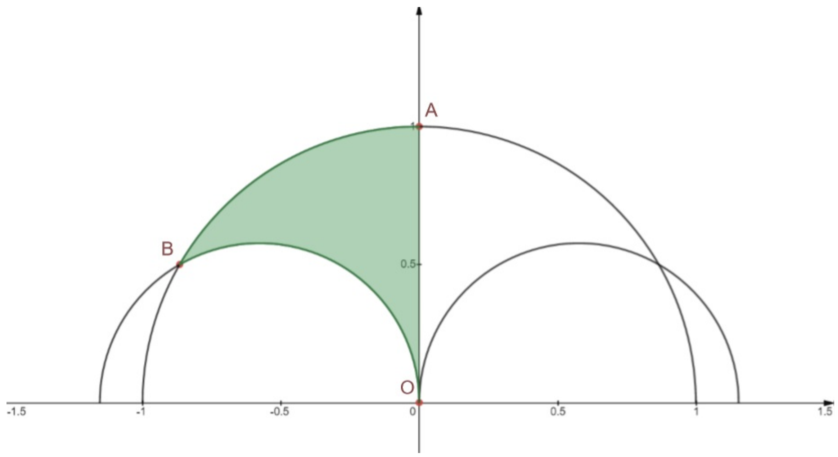



Figure 2.2: $30^\circ - 90^\circ - 0^\circ$ triangle

$\pi/\theta(S_0, S_1) = \infty$. By Theorem 9,

$$\langle S_i \mid S_i^2 = 1, (S_iS_j)^{k_{ij}} = 1\rangle$$

is a group presentation for $PDL_2(R_\sigma)$ under the mapping $S_i \mapsto \rho_i$.

Thus the map $\phi : (2\sigma, \infty) \to PDL_2(R_\sigma)$ defined by $\phi(s_j) = \rho_j$ for $j = 0, 1, 2$, (described in the beginning of this section) is an isomorphism. This proves the following result.

**Theorem 45.** *If $\sigma = 2$ or $\sigma = 3$, then $\phi : (2\sigma, \infty) \to PDL_2(R_\sigma)$ given by*

$$s_0 \mapsto \rho_0, s_1 \mapsto \rho_1 \textit{ and } s_2 \mapsto \rho_2$$

*is an isomorphism.*

# Chapter 3

# Dilinear Topograph

## 3.1 Divectors, dibases and pinwheels

In this section we define a “dilinear” variant of Conways’s topograph and prove basic facts about it. Assume $\sigma = 2$ or $\sigma = 3$.

**Definition 46.** The *dilinear topograph* is the incidence system of type $I = \{0, 1, 2\}$, consisting of:

- Faces (elements of type 2) are *primitive lax divectors* over $R_\sigma$, i.e., primitive divectors modulo $\pm\mathbf{1}$.
- Edges (elements of type 1) are *lax dibases*: unordered pairs of lax divectors generating $R_\sigma^2$ as an $R_\sigma$-module.
- Points (elements of type 0) are *lax pinwheels*: cyclically ordered $2\sigma$-tuples of lax divectors such that any adjacent pair forms a lax dibasis.

Two elements are incident if one contains another.

*Remark* 47.
- Note that $\pm(\sqrt{2},-1)$ and $\pm(-\sqrt{2},1)$ are the same as lax divectors. We prefer $\pm(\sqrt{2},-1)$ over $\pm(-\sqrt{2},1)$. More precisely, when writing a lax divector we prefer to make the first entry positive or zero. When the first entry is zero, we prefer the second entry to be positive. This means we prefer $\pm(0,1)$ over $\pm(0,-1)$.
- Let $(\pm\vec{v},\pm\vec{w})$ be an ordered lax dibasis. This means that the divectors $\vec{v}$ and $\vec{w}$ have opposite color and form the columns of a matrix in $DL_2(R_\sigma)$ of determinant $\pm 1$. Let $\mathcal{M}(\vec{v},\vec{w})$ denote this matrix and let $\mathcal{M}[\vec{v},\vec{w}]$ be its image in $PDL_2(R_\sigma)$.
- Given an ordered lax dibasis $(\pm\vec{v},\pm\vec{w})$ there are four associated matrices in $DL_2(R_\sigma)$: $\mathcal{M}(\vec{v},\vec{w})$, $\mathcal{M}(\vec{v},-\vec{w})$, $\mathcal{M}(-\vec{v},\vec{w})$ and $\mathcal{M}(-\vec{v},-\vec{w})$. These four matrices correspond to two distinct elements $\mathcal{M}[\vec{v},\vec{w}]$ and $\mathcal{M}[-\vec{v},\vec{w}]$ in $PDL_2(R_\sigma)$.

**Proposition 48.** *The group $PDL_2(R_\sigma)$ acts transitively on the set of lax dibases.*

*Proof.* Since the proof will be the same for any choice of order and signs of the divectors in a lax dibasis $\{\pm\vec{v},\pm\vec{w}\}$, it is enough to prove the result for the ordered dibasis $(\vec{v},\vec{w})$.

We need to show there exists a matrix $G \in PDL_2(R_\sigma)$ such that the "home dibasis" $((1,0),(0,1))$ is sent to $(\vec{v},\vec{w})$ under the action of $G$. That is, $G$ sends $(1,0)$ to $\vec{v}$ and $G$ sends $(0,1)$ to $\vec{w}$. Equivalently,

$$G\cdot\mathcal{M}\left[(1,0),(0,1)\right] = \mathcal{M}\left[\vec{v},\vec{w}\right].$$

Note that $\mathcal{M}\left[(1,0),(0,1)\right]$ is the identity in $PDL_2(R_\sigma)$. Thus $G = \mathcal{M}\left[\vec{v},\vec{w}\right]$. □

Theorem 39 proves that the dilinear group acts transitively on the set of primitive divectors, which implies that $PDL_2(R_\sigma)$ acts transitively on the set of faces of the dilinear topograph. The above result proves that $PDL_2(R_\sigma)$ acts transitively on the set edges.

**Proposition 49.** *If $\pm\vec{v}$ is a primitive lax divector then there exist infinitely many lax primitive divectors $\pm\vec{w}$ such that $\{\pm\vec{v}, \pm\vec{w}\}$ is a lax dibasis and these have the form*

$$\vec{w} = \vec{w_0} + n\sqrt{\sigma} \cdot \vec{v} \quad \textit{for } n \in \mathbb{Z}.$$

*Proof.* Let $\pm\vec{v} = (a, b\sqrt{\sigma})$ be a lax primitive red divector. Then $GCD(a, \sigma b) = 1$ in $\mathbb{Z}$. Assume that $\{\pm\vec{v} = (a, b\sqrt{\sigma}), \pm\vec{w} = (x\sqrt{\sigma}, y)\}$ is a lax dibasis (i.e. the regions corresponding to $\pm\vec{v}$ and $\pm\vec{w}$ share an edge in the topograph). Then $\begin{vmatrix} a & x\sqrt{\sigma} \\ b\sqrt{\sigma} & y \end{vmatrix} = \pm 1$ implies

$$(-\sigma b)x + (a)y = \pm 1.$$

Since $GCD(a, \sigma b) = 1$ this linear Diophantine equation has a solution. If $(x_0, y_0)$ is a solution of this equation then $\pm\vec{w_0} = (x_0\sqrt{\sigma}, y_0)$ is a blue divector such that $\{\pm\vec{v}, \pm\vec{w_0}\}$ is a lax dibasis. Since the other solutions to the above equation are of the form $(x_0 + na, y_0 + n \cdot \sigma b)$ for some $n \in \mathbb{Z}$ the vectors of the form $\pm\vec{w} = ((x_0 + na)\sqrt{\sigma}, y_0 + n \cdot \sigma b) = \underbrace{(x_0\sqrt{\sigma}, y_0)}_{\vec{w_0}} + n\sqrt{\sigma}\underbrace{(a, b\sqrt{\sigma})}_{\vec{v}}$ form a lax dibasis together with $\vec{v}$. □

This shows that every face is incident with infinitely many edges, and these form an endless line.

**Proposition 50.** *Let $\vec{v}$ and $\vec{w}$ be two divectors forming a dibasis. Then there are two pinwheels containing the lax dibasis $\{\pm\vec{v}, \pm\vec{w}\}$. These two pinwheels are:*

$$\sigma = 2 : \left(\pm\vec{v}, \pm\vec{w}, \pm(\vec{v} + \epsilon\vec{w}\sqrt{2}), \pm(\vec{w} + \epsilon\vec{v}\sqrt{2})\right)$$

$$\sigma = 3 : \left(\pm\vec{v}, \pm\vec{w}, \pm(\vec{v} + \epsilon\vec{w}\sqrt{3}), \pm(2\vec{w} + \epsilon\vec{v}\sqrt{3}), \pm(2\vec{v} + \epsilon\vec{w}\sqrt{3}), \pm(\vec{w} + \epsilon\vec{v}\sqrt{3})\right),$$

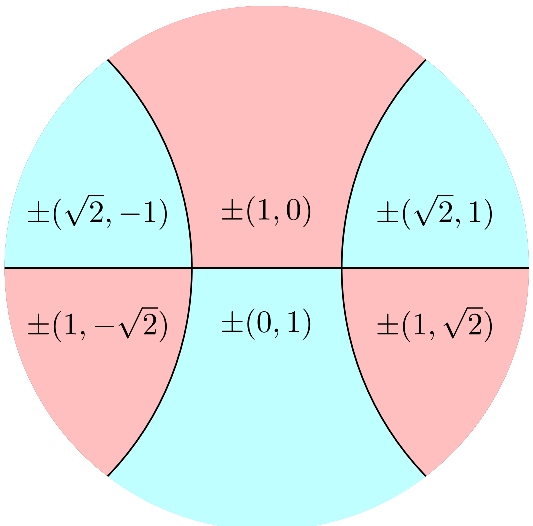
*where $\epsilon = 1$ and $\epsilon = -1$.*

*Proof.* Since the group $PDL_2(R_\sigma)$ acts transitively on the set of primitive lax divectors it suffices to assume $\pm\vec{v} = \pm(1,0)$ and $\pm\vec{w} = (0,1)$.

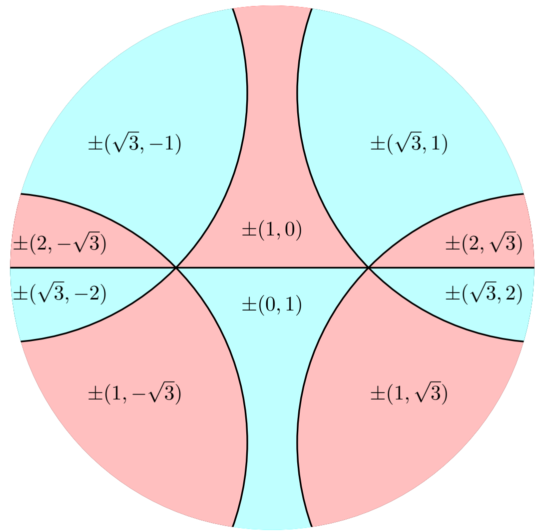
Consider the case $\sigma = 2$. Let $\{\pm(1,0), \pm(0,1), \pm\vec{u}, \pm\vec{t}\}$ be a pinwheel containing the lax dibasis $\{\pm(1,0), \pm(0,1)\}$. Note that $\vec{u}$ and $\pm(1,0)$ both form a lax dibasis with $\pm(0,1)$, and these dibases are adjacent lines in the dilinear topograph (they share the pinwheel as an endpoint). By Proposition 49, $\vec{u}$ has the form $\pm((1,0) + \epsilon\sqrt{2}(0,1))$ for $\epsilon \in \{-1,1\}$. The same reasoning shows that $\vec{t}$ has the form $\pm((0,1) + \epsilon'\sqrt{2}(1,0))$ for $\epsilon' \in \{-1,1\}$.

Since the divectors $\vec{u}$ and $\vec{t}$ are adjacent, they form the columns of a matrix in $DL_2(R_\sigma)$ of determinant $\pm1$. Thus $\begin{vmatrix} 1 & \epsilon'\sqrt{2} \\ \epsilon\sqrt{2} & 1 \end{vmatrix} = \pm1$, which implies $2\epsilon'\epsilon = 1 \pm 1$. Since $\epsilon, \epsilon' \in \{-1,1\}$, we must have $\epsilon = \epsilon'$. The case $\sigma = 3$ can be proved in a similar way. □

The above result proves that every edge is incident to two points. The chambers of the dilinear topograph are triples $(\pm\vec{v}, \mathcal{D}, \wp)$ with $\pm\vec{v}$ a lax divector, $\mathcal{D} = \{\pm\vec{v}, \pm\vec{w}\}$ a lax dibasis containing $\pm\vec{v}$ and $\wp$ a pinwheel containing $\mathcal{D}$.




(a) $\sigma = 2$




(b) $\sigma = 3$

Figure 3.1: Cell in the dilinear topograph containing the home lax dibasis.

*Remark* 51. By Proposition 50 the home lax dibasis $(\pm(1,0), \pm(0,1))$ is contained in exactly two pinwheels, which can be seen in Figure 3.1. The pinwheel we placed on the right side will be called the ***home lax pinwheel*** and it will be denoted $\wp_0$. More precisely,

$$\wp_0 = \begin{cases} (\pm(1,0), \pm(0,1), \pm(1,\sqrt{2}), \pm(\sqrt{2},1)) & \sigma = 2 \\ (\pm(1,0), \pm(0,1), \pm(1,\sqrt{3}), \pm(\sqrt{3},2), \pm(2,\sqrt{3}), \pm(\sqrt{3},1)) & \sigma = 3. \end{cases}$$

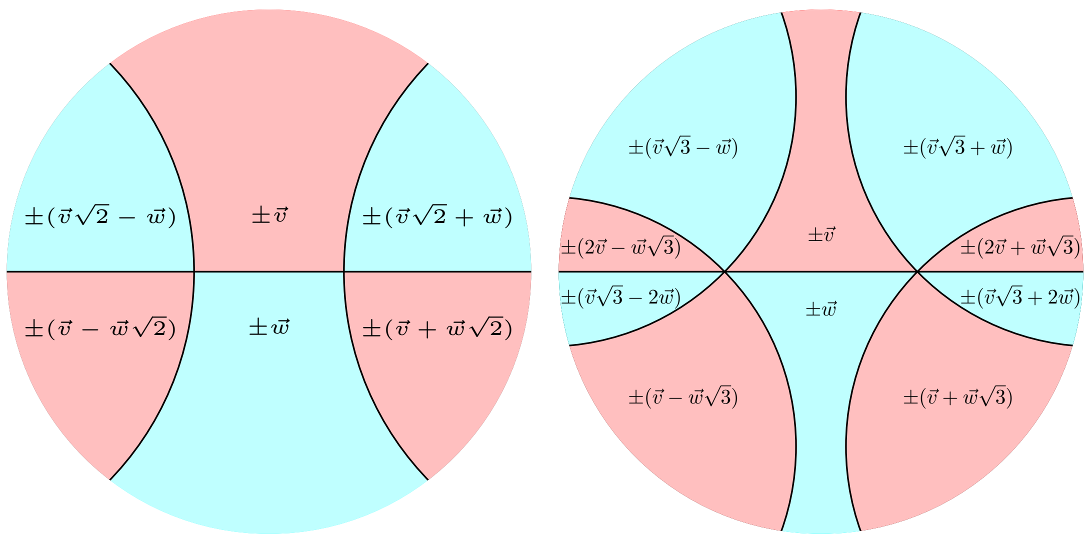
The flag $\mathcal{F}_0 = (v_0 = \pm(1,0), \mathcal{D}_0 = (\pm(1,0), \pm(0,1)), \wp_0)$ containing the home lax pinwheel will be called the ***home flag***.

By Proposition 50 the ordered dibasis $\mathcal{D} = (\pm v, \pm w)$ is contained in exactly two pinwheels starting with $\pm\vec{v}$. In the $\sigma = 2$ case, $\wp^+ = \big(\pm\vec{v}, \pm\vec{w}, \pm(\vec{v} + \vec{w}\sqrt{2}), \pm(\vec{v}\sqrt{2} + \vec{w})\big)$

and $\wp^- = \left(\pm\vec{v}, \pm\vec{w}, \pm(\vec{v} - \vec{w}\sqrt{2}), \pm(\vec{v}\sqrt{2} - \vec{w})\right)$. Similarly, in the $\sigma = 3$ case,

$$\wp^+ = \left(\pm\vec{v}, \pm\vec{w}, \pm(\vec{v} + \vec{w}\sqrt{3}), \pm(\vec{v}\sqrt{3} + 2\vec{w}), \pm(2\vec{v} + \vec{w}\sqrt{3}), \pm(\vec{v}\sqrt{3} + \vec{w})\right)$$

$$\text{and } \wp^- = \left(\pm\vec{v}, \pm\vec{w}, \pm(\vec{v} - \vec{w}\sqrt{3}), \pm(\vec{v}\sqrt{3} - 2\vec{w}), \pm(2\vec{v} - \vec{w}\sqrt{3}), \pm(\vec{v}\sqrt{3} - \vec{w})\right).$$




(a) $\sigma = 2$ (b) $\sigma = 3$

Figure 3.2: Cell in the dilinear topograph containing the lax dibasis $(\pm v, \pm w)$ with the pinwheel $\wp^+$ placed on the right side.

**Proposition 52.** *The group $PDL_2(R_\sigma)$ acts transitively on the set of chambers of the dilinear topograph.*

*Proof.* Let $\mathcal{F} = (\pm\vec{v}, \mathcal{D} = (\pm\vec{v}, \pm\vec{w}), \wp)$ be a chamber (i.e a flag of type $I = \{0, 1, 2\}$). We need to show there exists a matrix $G \in PDL_2(R_\sigma)$ such that the flag $\mathcal{F}$ is sent to the home flag $\mathcal{F}_0 = (v_0, \mathcal{D}_0, \wp_0)$ under the action of $G$. It is easy to check that the matrix $\mathcal{M}[\vec{v}, \vec{w}]^{-1}$ sends $\pm\vec{v}$ to $v_0$ and $\mathcal{D}$ to $\mathcal{D}_0$. Moreover, it must send $\wp$ to one of the endpoints of the edge $\mathcal{D}$ (since the group action preserves incidence). More precisely, it sends $\wp$ to $\wp_0$ or the other endpoint $\wp_0^-$. In the first case, $G = \mathcal{M}[\vec{v}, \vec{w}]^{-1}$ is the desired matrix. In the second case, $G = \rho_0 \mathcal{M}[\vec{v}, \vec{w}]^{-1}$. Here $\rho_0$ is the reflection matrix defined in Section 2.3.

□

In the beginning of this section we defined the dilinear topograph as an incidence system. The following result shows that it is an incidence geometry.

**Proposition 53.** *Every maximal flag of the dilinear topograph is a chamber.*

*Proof.* A maximal flag is a flag not properly contained in any other flag. That means we need to show that every partial flag can be completed to a chamber. More precisely, we need to make sure that the following hold:

(a) Every vertex is contained in a chamber.

(b) Every edge is contained in a chamber.

(c) Every face is contained in a chamber.

(d) Every pair $(v, e)$ of a vertex incident with an edge can be completed to a chamber.

(e) Every pair $(v, f)$ of a vertex incident with a face can be completed to a chamber.

(f) Every pair $(e, f)$ of an edge incident with a face can be completed to a chamber.

To see why (a) is true note that a vertex $v$ is a pinwheel. We can take any lax dibasis $e$ contained in the pinwheel $v$ and obtain a (nonmaximal) flag $(v, e)$. Then take any primitive lax divector within $e$. That gives a face $f$ for which $(v, e, f)$ is a chamber.

Since an edge is a lax dibasis, (b) follows from Proposition 50. By Proposition 49, any primitive lax divector can be completed to a lax dibasis. Since a face $f$ is a primitive lax divector, (c) follows from Proposition 49 and Proposition 50. The claims (d), (e), (f) are easy to check. □

## 3.2 Coxeter geometry

Assume $\sigma = 2$ or $\sigma = 3$. In the previous section we showed that the dilinear topograph is an incidence geometry. Let $X^0$ denote its set of points, $X^1$ the set of edges, and $X^2$ the set of faces. Let $X = X^0 \sqcup X^1 \sqcup X^2$. By a slight abuse of notation, we will also write $X$ to denote both the set of elements of the incidence geometry and the incidence geometry itself.

Let $(W, S)$ denote the Coxeter system with $W = (2\sigma, \infty)$ and $S = \{s_0, s_1, s_2\}$. Recall that $W^i$ denotes the parabolic subgroup of $W$ generated by $s \in S \setminus \{s_i\}$. Let $X_W = \Gamma(W, \{W^0, W^1, W^2\})$ denote its coset incidence geometry. By slight abuse of notation, we also let $X_W$ denote the underlying set of the geometry. That is,

$$X_W = \bigsqcup_{i=0}^{2} X_W^i \text{ where } X_W^i = W/W^i.$$

We proved in Section 2.3 that $\phi : (2\sigma, \infty) \to PDL_2(R_\sigma)$ given by $\phi(s_j) = \rho_j$ for $j = 0, 1, 2$, is an isomorphism. Recall that

$$\rho_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \rho_1 = \begin{bmatrix} -1 & 0 \\ \sqrt{\sigma} & 1 \end{bmatrix}, \rho_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

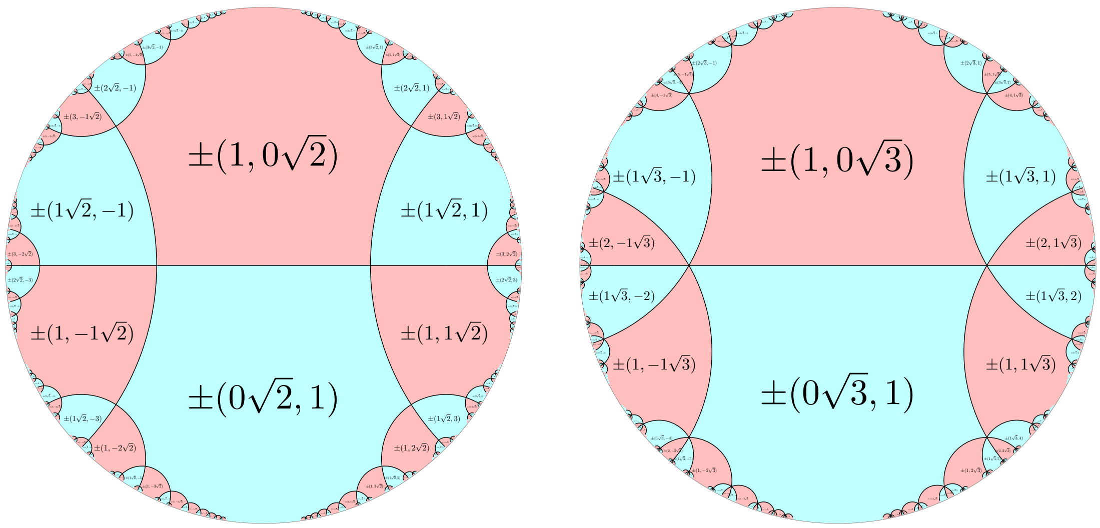



Figure 3.3: The geometry of primitive lax divectors, lax dibases, and pinwheels for $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$, respectively.

Recall that $\upsilon_0 = \pm(1,0)$ denotes the home face, $\mathcal{D}_0 = (\pm(1,0), \pm(0,1))$ the home edge and $\wp_0$ the home vertex. The following lemma gives a description of the stabilizer of the home vertex/edge/face.

**Lemma 54.** *Let $G = PDL_2(R_\sigma)$ and let $G^i$ denote the parabolic subgroup of $G$ generated by $\rho \in \{\rho_0, \rho_1, \rho_2\} \setminus \{\rho_i\}$. The following hold:*

*(a)* $Stab_G(\upsilon_0) = G^2$

*(b)* $Stab_G(\mathcal{D}_0) = G^1$

*(c)* $Stab_G(\wp_0) = G^0$.

*Proof.* Let $\gamma \in G$. If $\gamma = \begin{bmatrix} u\sqrt{\sigma} & x \\ v & y\sqrt{\sigma} \end{bmatrix}$ it is easy to see that it can't be in $Stab_G(\upsilon_0)$.

If $\gamma = \begin{bmatrix} u & x\sqrt{\sigma} \\ v\sqrt{\sigma} & y \end{bmatrix} \in Stab_G(\upsilon_0)$ then $\begin{bmatrix} u & x\sqrt{\sigma} \\ v\sqrt{\sigma} & y \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x\sqrt{\sigma} \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since $\gamma$ has determinant $\pm 1$ we get $\gamma = \begin{bmatrix} 1 & 0 \\ v\sqrt{\sigma} & 1 \end{bmatrix}$, which we can write as $(r_0 r_1)^v$. We have $Stab_G(\upsilon_0) \subseteq \langle r_0, r_1 \rangle = G^2$ and we can check that $r_0, r_1 \in Stab_G(\upsilon_0)$. Thus (a) is proved. Similar arguments can be used to prove (b) and (c). □

**Theorem 55.** *The geometry $X$ is isomorphic to the coset geometry $X_W$ of the Coxeter group* $(2\sigma, \infty)$.

*Proof.* We need to show there exist an isomorphism of incidence systems $\beta : X_W \to X$. Since the set $X_W$ is the disjoint union of the sets $X_W^0, X_W^1$ and $X_W^2$, the map $\beta$ can be described by a triple of maps $\beta^0 : X_W^0 \to X^0$, $\beta^1 : X_W^1 \to X^1$ and $\beta^2 : X_W^2 \to X^2$.

Let $x^0 = \wp_0$ (home vertex), $x^1 = \mathcal{D}_0$ (home edge) and $x^2 = \upsilon_0$ (home face). Let $i \in \{0, 1, 2\}$. Given a coset $wW^i \in X_W^i$, define $\beta^i : X_W^i \to X^i$ by

$$\beta^i(wW^i) = \phi(w)x^i.$$

Since $G = PDL_2(R_\sigma)$ acts transitively on the set of vertices and $\phi$ is surjective it follows that $\beta^i$ is surjective. Now we want to show $\beta^i$ is injective. Assume $\beta^i(w_1W^i) = \beta^i(w_2W^i)$ for some cosets $w_1W^i, w_2W^i \in X_W^i$. That is, $\phi(w_1)x^i = \phi(w_2)x^i$, which is equivalent to $\phi(w_2^{-1}w_1)x^i = x^i$. In other words, $\phi(w_2^{-1}w_1) \in Stab_G(x^i) = G^i = \phi(W^i)$, which implies $w_1W^i = w_2W^i$. Thus $\beta^i$ is a bijection.

Clearly $\beta$ preserves the type. It remains to check that $\beta$ preserves incidence. That is, if two cosets of $W$ are incident then their images under $\beta$ are also incident. Let

$s, t \in \{0, 1, 2\}$ with $s < t$. If $w'W^s$ is incident to $w^{''}W^t$ then $w'W^s \cap w^{''}W^t \neq \emptyset$. Let $w \in w'W^s \cap w^{''}W^t$. That is, $wW^s = w'W^s$ and $wW^t = w^{''}W^t$. Then $\beta^s(w'W^s) = \beta^s(wW^s) = \phi(w)x^s$ and $\beta^t(w^{''}W^t) = \beta^t(wW^t) = \phi(w)x^t$. Since $x^t \subset x^s$, we also have $\phi(w)x^t \subset \phi(w)x^s$. This shows that $\phi(w')x^s$ is incident to $\phi(w^{''})x^t$. □

We were able to show that not only is there an isomorphism of groups - from the Coxeter group to the arithmetic group (dilinear group) - but there is an isomorphism of geometries - from the Coxeter geometry to the geometry of arithmetic flags. The next two chapters include applications to number theory.

# Chapter 4

# Binary quadratic diforms

## 4.1 Diforms and their connection to pairs of BQFs

We introduce a special type of non-integral BQFs defined on the set of divectors and show how they are connected to pairs of (integral) BQFs obtained by restricting to red/blue divectors.

**Definition 56.** Let $\sigma$ be a square-free positive integer. A *binary quadratic diform* (BQD) is a function $Q : R_\sigma^{\text{di}} \to \mathbb{Z}$ of the form

$$Q(x, y) = ax^2 + b\sqrt{\sigma}xy + cy^2, \text{ where } a, b, c \in \mathbb{Z}.$$

We define the discriminant of $Q$ by $\Delta(Q) = \sigma(b^2\sigma - 4ac)$.

We restrict $(x, y)$ to be a divector in $R_\sigma^2$, so the values of $Q$ are integers. Restricting $Q$ to red and blue divectors yields a pair $Q_{\text{red}}, Q_{\text{blue}}$ of BQFs over $\mathbb{Z}$ of discriminant $\Delta$; explicitly,

$$Q_{\text{red}}(u,v) := Q(u, v\sqrt{\sigma}) = au^2 + b\sigma uv + c\sigma v^2,$$

$$Q_{\text{blue}}(u,v) := Q(u\sqrt{\sigma}, v) = a\sigma u^2 + b\sigma uv + cv^2.$$

**Definition 57.** We say $Q$ is a *primitive* BQD if both $Q_{red}$ and $Q_{blue}$ are primitive BQFs. Equivalently, $\text{GCD}(a, \sigma b, c) = 1$ and $\sigma$ divides neither $a$ nor $c$.

Define another BQF of discriminant $\Delta$,

$$A_\Delta(u,v) = \begin{cases} \sigma u^2 - \frac{\Delta}{4\sigma} v^2 & \text{if } \Delta\sigma^{-1} \equiv 0 \bmod 4; \\ \sigma u^2 + \sigma uv - \frac{\Delta - \sigma^2}{4\sigma} v^2 & \text{if } \Delta\sigma^{-1} \not\equiv 0 \bmod 4. \end{cases}$$

**Lemma 58.** *If $Q$ is a primitive BQD of discriminant $\Delta$, then $A_\Delta$ is primitive too.*

*Proof.* Assume $\text{GCD}(a, \sigma b, c) = 1$ and $\sigma$ divides neither $a$ nor $c$.

Let $\sigma = 2$. Assume $\Delta\sigma^{-1} \equiv 0 \bmod 4$. Then $\text{GCD}(\sigma, -\frac{\Delta}{4\sigma}) = \text{GCD}(2, ac - \frac{b^2}{2})$ equals 1 or 2. Since $2 \nmid ac$, $\text{GCD}(2, ac - \frac{b^2}{2}) = 2$ implies $\frac{b^2}{2}$ is an odd integer. We get a contradiction with $b^2 \equiv 0, 1 \bmod 4$ for any $b \in \mathbb{Z}$ . Thus we must have $\text{GCD}(\sigma, -\frac{\Delta}{4\sigma}) = 1$. Now assume $\Delta\sigma^{-1} \not\equiv 0 \bmod 4$. When $\text{GCD}(\sigma, -\frac{\Delta-\sigma^2}{4\sigma}) = \text{GCD}(2, \frac{1-b^2}{2} - ac) = 2$ we get a contradiction with $1 - b^2 \equiv 0, 1 \bmod 4$. This shows $A_\Delta$ is primitive.

Let $\sigma = 3$. Assume $\Delta\sigma^{-1} \equiv 0 \bmod 4$. Then $\text{GCD}(\sigma, -\frac{\Delta}{4\sigma}) = \text{GCD}(3, ac - \frac{3b^2}{4})$ equals 1 or 3. But $\text{GCD}(3, ac - \frac{3b^2}{4}) = 3$ implies $ac \equiv \frac{3b^2}{4} \bmod 3$. We get a contradiction since $3 \nmid ac$ and $\frac{3b^2}{4} \equiv 0 \bmod 3$. Thus we must have $\text{GCD}(\sigma, -\frac{\Delta}{4\sigma}) = 1$. This shows $A_\Delta$ is primitive in the case $\Delta\sigma^{-1} \equiv 0 \bmod 4$. Similarly, it can be checked that $A_\Delta$ is primitive in the case $\Delta\sigma^{-1} \not\equiv 0 \bmod 4$. □

**Notation 59.** *Write* $\mathrm{Cl}(\Delta)$ *for the group of* $SL_2(\mathbb{Z})$*-equivalence classes of primitive BQFs of discriminant* $\Delta$*, following Bhargava [7, Theorem 1]. If* $Q$ *is a BQF of discriminant* $\Delta$*, write* $[Q]$ *for its* $SL_2(\mathbb{Z})$*-equivalence class.*

A binary quadratic form $Q(x,y) = ax^2+bxy+cy^2$ is called *ambiguous* if its first coefficient $a$ divides its middle coefficient $b$. An *ambiguous class* is one which contains an ambiguous form. The primitive ambiguous classes are those which are self-inverse under composition. Since $A_\Delta$ is an ambiguous form, its class in $\mathrm{Cl}(\Delta)$ satisfies $[A_\Delta]^2 = 1$. We give another characterization of the class of $A_\Delta$ in the following lemma. We will show that $[A_\Delta]$ is the unique class in $\mathrm{Cl}(\Delta)$ which represents $\sigma$, when $\sigma \mid \Delta$.

**Lemma 60.** *If* $Q$ *is a BQF of discriminant* $\Delta$ *that represents* $\sigma$*, and* $\sigma \mid \Delta$*, then* $[Q] = [A_\Delta]$.

*Proof.* If $Q$ represents $\sigma$, then $[Q] = [\sigma u^2 + buv + cv^2]$ for some $b, c \in \mathbb{Z}$. Since $\sigma$ is a square-free integer and $\sigma \mid \Delta = b^2 - 4\sigma c$ we have that $b$ is a multiple of $\sigma$ too. Thus $[Q] = [\sigma u^2 + \sigma\beta uv + cv^2]$ for some $\beta$. Since this is an ambiguous class it must be equal to either $[\sigma u + kv^2]$ or $[\sigma u^2 + uv + kv^2]$ for some $k \in \mathbb{Z}$, depending on the discriminant.

If $[Q] = [\sigma u + kv^2]$ we have $\Delta = -4\sigma k$ (i.e. $k = -\Delta/4\sigma$) and $\Delta\sigma^{-1} \equiv 0 \bmod 4$. If $[Q] = [\sigma u^2 + \sigma uv + kv^2]$, then $\Delta = \sigma^2 - 4\sigma k$ (i.e. $k = -(\Delta - \sigma^2)/(4\sigma)$) and $\Delta\sigma^{-1} \equiv \sigma - 4k \not\equiv 0 \bmod 4$. Thus $[Q] = [A_\Delta]$. □

**Notation 61.** *Let* $SDL_2^+(R_\sigma)$ *be the subgroup of* $DL_2^+(R_\sigma)$ *consisting of matrices of determinant one.*

**Definition 62.** We say that two diforms $Q, Q'$ are $SDL_2^+(R_\sigma)$-*equivalent* if there exists $\eta \in SDL_2^+(R_\sigma)$ satisfying $Q'(\vec{v}) = Q(\eta \cdot \vec{v})$ for all divectors $\vec{v}$. We write $[Q]_\sigma = [Q']_\sigma$ when the diforms $Q$ and $Q'$ are $SDL_2^+(R_\sigma)$-equivalent.

**Proposition 63.** $[Q]_\sigma = [Q']_\sigma$ *implies* $[Q_{\text{red}}] = [Q'_{\text{red}}]$ *and* $[Q_{\text{blue}}] = [Q'_{\text{blue}}]$.

*Proof.* Let $g = diag(1, \sqrt{\sigma})$. Then

$$Q_{\text{red}}(u, v) = Q(u, \sqrt{\sigma}v) = Q(g \cdot (u, v)).$$

Let $M$ denote the Gram matrix of the quadratic form $Q$, and $M_{\text{red}}$ the Gram matrix of the quadratic form $Q_{\text{red}}$ (viewing them as quadratic forms $\mathbb{R}^2 \to \mathbb{R}$). Then we have $Q_{\text{red}}(u, v) = (u\ v) \cdot M_{\text{red}} \cdot \binom{u}{v}$ and $Q(g \cdot (u, v)) = \left(g \cdot \binom{u}{v}\right)^t \cdot M \cdot g \cdot \binom{u}{v}$. Note that $g^t = g$. Thus

$$M_{\text{red}} = g \cdot M \cdot g.$$

Now assume $[Q]_\sigma = [Q']_\sigma$. That is, there exists $\eta \in SDL_2^+(R_\sigma)$ such that $Q'(\vec{w}) = Q(\eta \cdot \vec{w})$ for all divectors $\vec{w}$. The Gram matrix of $Q'$ is

$$M' = \eta^t M \eta.$$

The Gram matrix of $Q'_{\text{red}}$ is

$$M'_{\text{red}} = gM'g = g(\eta^t M \eta)g = g\eta^t g^{-1}(gMg)g^{-1}\eta g = g\eta^t g^{-1} M_{\text{red}} g^{-1}\eta g.$$

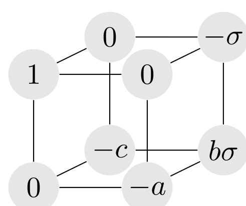
Thus $M'_{\text{red}} = (g^{-1}\eta g)^t M_{\text{red}} g^{-1}\eta g$. So to show that $Q'_{\text{red}}$ is $SL_2(\mathbb{Z})$-equivalent to $Q_{\text{red}}$, we must check that $g^{-1}\eta g \in SL_2(\mathbb{Z})$. Let $\eta = \begin{pmatrix} u & x\sqrt{\sigma} \\ v\sqrt{\sigma} & y \end{pmatrix} \in DL_2^+(R_\sigma)$. Then

$$g^{-1}\eta g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\sigma}} \end{pmatrix} \begin{pmatrix} u & x\sqrt{\sigma} \\ v\sqrt{\sigma} & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\sigma} \end{pmatrix} = \begin{pmatrix} u & x\sigma \\ v & y \end{pmatrix} \in SL_2(\mathbb{Z}). \qquad \square$$

The following result shows how the pair $Q_{\text{red}}, Q_{\text{blue}}$ of BQFs can be related through $A_\Delta$ whenever $\sigma \mid \Delta$.

**Theorem 64.** *Given a primitive BQD $Q(x, y) = ax^2 + b\sqrt{\sigma}xy + cy^2$ of discriminant $\Delta$, one has $[Q_{\text{red}}] = [A_\Delta] \cdot [Q_{\text{blue}}]$ in $\mathrm{Cl}(\Delta)$. Conversely, if $Q_1, Q_2$ are primitive BQFs of discriminant $\Delta$, and $\sigma \mid \Delta$, and $[Q_1] = [A_\Delta] \cdot [Q_2]$, there exists a primitive BQD $Q$ such that $[Q_{\text{red}}] = [Q_1]$ and $[Q_{\text{blue}}] = [Q_2]$.*

*Proof.* The identity $[Q_{\text{red}}] = [A_\Delta] \cdot [Q_{\text{blue}}]$ can be proved using a Bhargava cube.




Let $(M_i, N_i)$ be the partition of this cube into a pair of two-by-two matrices, in a front-back, left-right, and top-bottom fashion according to whether $i = 1, 2, 3$ respectively, as in [7, §2.1]. From these matrices, Bhargava constructs a triple of BQFs given by $Q_i(u, v) = -\det(M_i u - N_i v)$. More precisely,

$$Q_1(u, v) = au^2 + b\sigma uv + c\sigma v^2;$$

$$Q_2(u, v) = cu^2 + b\sigma uv + a\sigma v^2;$$

$$Q_3(u, v) = \sigma u^2 + b\sigma uv + acv^2.$$

Since $Q(x,y) = ax^2 + b\sqrt{\sigma}xy + cy^2$ if a primitive diform of discriminant $\Delta$, the BQFs $Q_1, Q_2$ and $Q_3$ are primitive and have discriminant $\Delta$ also. By [7, Theorem 1], we have $[Q_1] \cdot [Q_2] \cdot [Q_3] = 1$ in $\mathrm{Cl}(\Delta)$. Observe that $Q_1$ is precisely $Q_{\mathrm{red}}$. Next, observe that $Q_2$ is related to $Q_{\mathrm{blue}}$ by switching $u$ and $v$; it follows that $[Q_2] = [Q_{\mathrm{blue}}]^{-1}$. By Lemma 60, $[Q_3] = [A_\Delta]$. Since $[A_\Delta]^2 = 1$, we have

$$[Q_{\mathrm{red}}] = [A_\Delta] \cdot [Q_{\mathrm{blue}}] \text{ and } [Q_{\mathrm{blue}}] = [A_\Delta] \cdot [Q_{\mathrm{red}}].$$

For the converse, suppose that $Q_1$ and $Q_2$ are primitive BQFs of discriminant $\Delta$, $\sigma \mid \Delta$, and $[Q_2] = [A_\Delta] \cdot [Q_1]$. If $Q_1$ is any BQF of discriminant $\Delta$, then we want to show a primitive BQD $Q$ with $[Q_{\mathrm{red}}] = [Q_1]$ can be defined. Since $[Q_2] = [A_\Delta] \cdot [Q_1]$, the identity $[Q_{\mathrm{blue}}] = [A_\Delta] \cdot [Q_{\mathrm{red}}]$ will imply that $[Q_{\mathrm{blue}}] = [Q_2]$.

Write $Q_1(u,v) = \alpha u^2 + \beta uv + \gamma v^2$, so $\sigma \mid \beta^2 - 4\alpha\gamma$. If $\sigma \mid \gamma$, then $\sigma \mid \beta$, and $Q_1 = Q_{\mathrm{red}}$ for the diform

$$Q(x,y) = \alpha x^2 + \beta\sigma^{-1}\sqrt{\sigma}xy + \gamma\sigma^{-1}y^2.$$

If $\sigma \nmid \gamma$, then there exists an integer $v$ satisfying the congruence $\alpha + \beta v + \gamma v^2 \equiv 0 \bmod \sigma$. One may check this working one prime divisor of $\sigma$ at a time; the quadratic formula applies for odd prime divisors. Modulo two, $2 \mid \sigma \mid \beta^2 - 4\alpha\gamma$ implies that $\beta$ is even and the congruence has a solution. Hence $Q_1(1,v) \equiv 0 \bmod \sigma$.

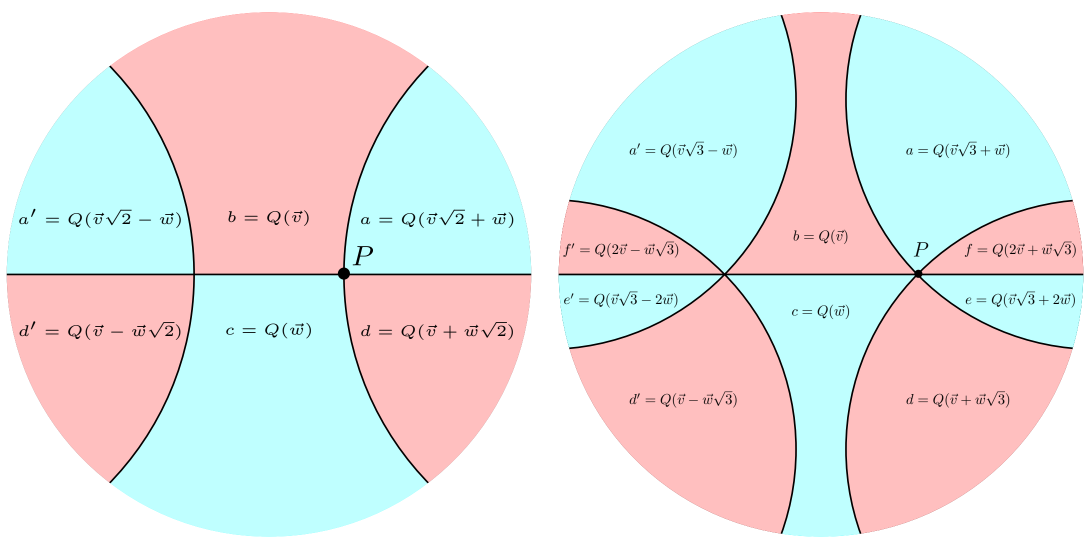
Since $Q_1$ represents a multiple of $\sigma$, $Q_1$ is equivalent to a form $au^2 + \beta' uv + c\sigma v^2$. The fact that $\sigma$ divides the discriminant implies $\beta' = b\sigma$ for some $b \in \mathbb{Z}$. Thus, whether $\sigma$ divides $\gamma$ or not, $[Q_1] = [Q_{\mathrm{red}}]$ for some diform $Q$. $\square$

**Corollary 65.** *Assume $\sigma \mid \Delta$. The map $Q \mapsto (Q_{\text{red}}, Q_{\text{blue}})$ yields a surjective function from the set of $SDL_2^+(R_\sigma)$-equivalence classes of BQFs of discriminant $\Delta$ to the set of ordered pairs $([Q_1], [Q_2])$ in $\mathrm{Cl}(\Delta)$ satisfying $[Q_1] = [A_\Delta] \cdot [Q_2]$.*

## 4.2 Dilinear topographs of BQDs

Here we return to the assumption that $\sigma = 2$ or $\sigma = 3$. The *topograph* of a BQD $Q$ is obtained by replacing each primitive lax divector by the corresponding value of $Q$.

### 4.2.1 Arithmetic progression rule




(a) $\sigma = 2$ (b) $\sigma = 3$

Figure 4.1

We describe here some properties that can be used to quickly obtain the topograph of a BQD, beginning with only the values around a vertex. The *polarization identity*

$$Q(\vec{v}_1 + \vec{v}_2) + Q(\vec{v}_1 - \vec{v}_2) = 2\left(Q(\vec{v}_1) + Q(\vec{v}_2)\right),$$

holds for any quadratic form $Q$ and any two-dimensional vectors $\vec{v}_1, \vec{v}_2$. This formula also tells us that the sequence

$$Q(\vec{v}_1 - \vec{v}_2),\ Q(\vec{v}_1) + Q(\vec{v}_2),\ Q(\vec{v}_1 + \vec{v}_2)$$

is an arithmetic progression with step size given by the symmetric bilinear form associated to the quadratic form $Q$. That is, $B_Q(\vec{v}_1, \vec{v}_2) = Q(\vec{v}_1+\vec{v}_2) - Q(\vec{v}_1) - Q(\vec{v}_2)$. Conway verifies and uses the polarization identity in [2] to obtain his Arithmetic Progression Rule for the topograph of a BQF.

**Theorem 66.** *At every cell in the topograph of $Q$, as in Figure 4.1, one finds arithmetic progressions as below.*

$\sigma = 2$: *The triples $(a', 2b+c, a)$ and $(d', b+2c, d)$ are arithmetic progressions of the same step size.*

$\sigma = 3$: *The triples $(a', 3b + c, a)$ and $(d', b + 3c, d)$ are arithmetic progressions of the same step size $\delta$ and the triples $(f', 4b + 3c, f)$ and $(e', 3b + 4c, e)$ are arithmetic progressions of the same step size $2\delta$.*

*Proof.* In both cases $\sigma = 2, 3$, the integers $a', b, c, a$ of a cell arise as values of $Q$ as displayed in Figure 4.1. More precisely,

$$a' = Q(\sqrt{\sigma}\vec{v} - \vec{w}),\ b = Q(\vec{v}),\ c = Q(\vec{w}),\ a = Q(\sqrt{\sigma}\vec{v} + \vec{w}).$$

Note that $Q(\sqrt{\sigma}\vec{v}) = \sigma Q(\vec{v}) = \sigma b$, and $Q(\sqrt{\sigma}\vec{w}) = \sigma Q(\vec{w}) = \sigma c$. Then the polarization identity implies that the sequence

$$a' = Q(\sqrt{\sigma}\vec{v} - \vec{w}),\ \sigma b + c = Q(\sqrt{\sigma}\vec{v}) + Q(\vec{w}),\ a = Q(\sqrt{\sigma}\vec{v} + \vec{w})$$

is an arithmetic progression with step size $\delta := B_Q(\sqrt{\sigma}\vec{v}, \vec{w})$. Moreover, the sequence

$$d' = Q(\vec{v} - \sqrt{\sigma}\vec{w}),\ b + \sigma c = Q(\vec{v}) + Q(\sqrt{\sigma}\vec{w}),\ d = Q(\vec{v} + \sqrt{\sigma}\vec{w})$$

is an arithmetic progression with step size $\delta' = B_Q(\vec{v}, \sqrt{\sigma}\vec{w})$. Note that $\delta = \delta'$. Hence $(a', \sigma b + c, a)$ and $(d', b + \sigma c, d)$ are arithmetic progressions of the same step size.

Similarly, when $\sigma = 3$, the polarization identity can be used again to show that $f' = Q(2\vec{v} - \vec{w}\sqrt{3})$, $4b + 3c = Q(2\vec{v}) + Q(\vec{w}\sqrt{3})$, $f = Q(2\vec{v} + \sqrt{3}\vec{w})$ and $e' = Q(\vec{v}\sqrt{3} - 2\vec{w})$, $3b + 4c = Q(\vec{v}\sqrt{3}) + Q(2\vec{w})$, $e = Q(\vec{v}\sqrt{3} + 2\vec{w})$ are arithmetic progressions of the same step size $2\delta$. $\square$

We draw an arrow on each edge to represent the direction of increasing progressions, or a circle if all progressions are constant. Figure 4.2 displays an example. The climbing principle is the same as Conway's.

**Corollary 67.** *Suppose $b$, $c$ in Figure 4.1 are positive. Moreover, assume that the step size $\delta$ of the arithmetic progressions is positive. Then the other values around the vertex $P$ are also positive, and the edges that emerge from $P$ all point away from $P$.*

With the same notation as in Figure 4.1 (same as in the vertex diagrams given below) we obtain linear relations among the values around a vertex in the topograph.

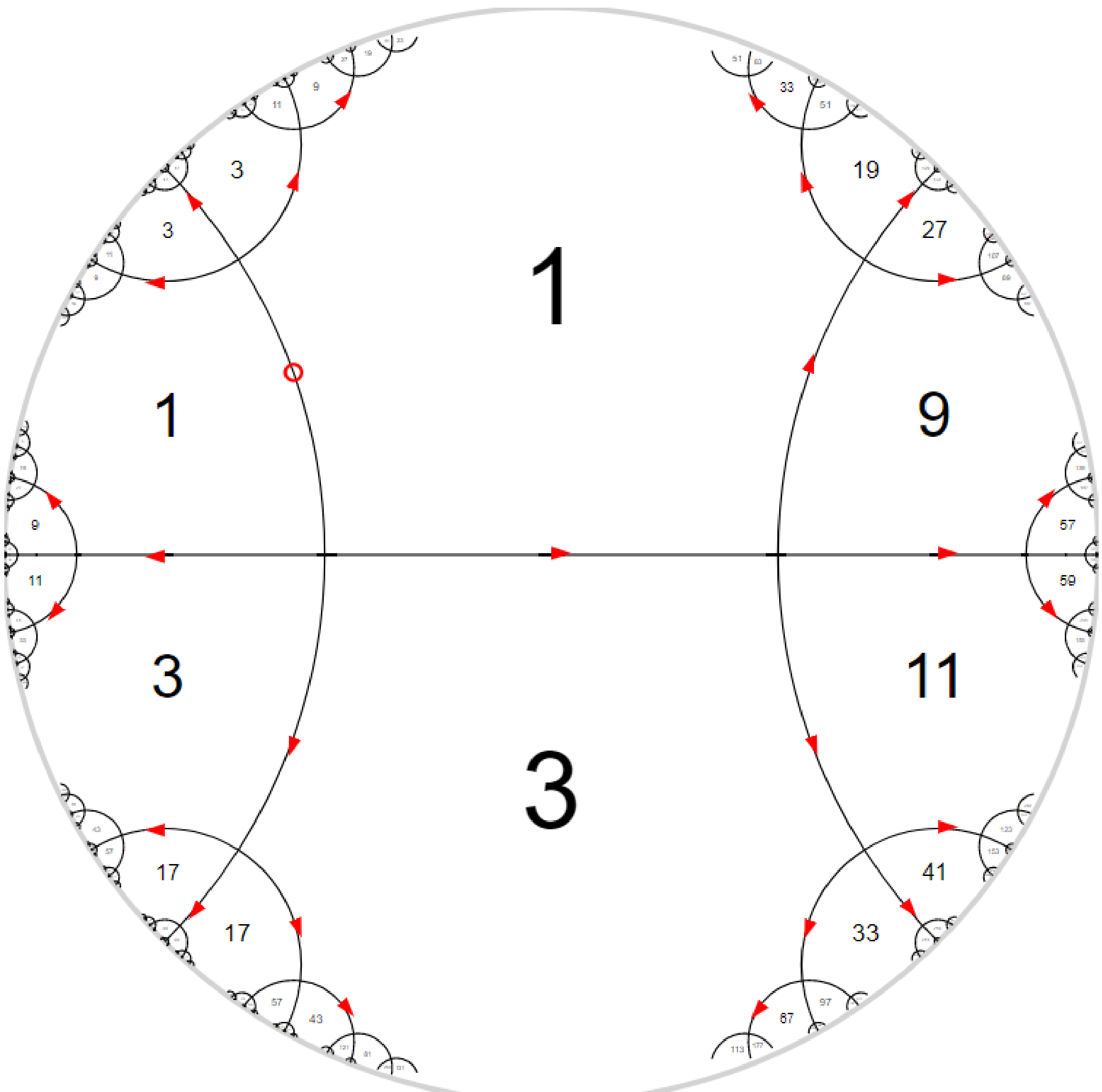
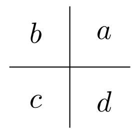
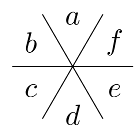



Figure 4.2: Topograph for the binary quadratic diform $Q(x, y) = x^2 - 2\sqrt{2}xy + 3y^2$.






**Corollary 68.** *When* $\sigma = 2$*,* $a + c = b + d$*. When* $\sigma = 3$*,* $a + d = b + e = c + f$ *and also* $a + c + e = b + d + f$*.*

*Proof.* When $\sigma = 2$, the triples $(a', 2b+c, a)$ and $(d', b+2c, d)$ are arithmetic progressions of the same step size $\delta$. Since $\delta = a - (2b + c) = d - (b + 2c)$ we get the relation $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$.

When $\sigma = 3$, the triples $(f', 4b + 3c, f)$ and $(e', 3b + 4c, e)$ are arithmetic progressions of the same step size $2\delta$. Since $2\delta = f - (4b + 3c) = e - (3b + 4c)$ we get the relation $\mathbf{c} + \mathbf{f} = \mathbf{b} + \mathbf{e}$.

Moreover, the triples $(a', 3b + c, a)$ and $(d', b + 3c, d)$ are arithmetic progressions of the same step size $\delta = a - (3b + c) = d - (b + 3c)$. This gives the relation $\mathbf{a} = \mathbf{d} + \mathbf{2}(\mathbf{b} - \mathbf{c})$. The relation $\mathbf{a} + \mathbf{d} = \mathbf{b} + \mathbf{e}$ follows from $2\delta = a - (3b + c) + d - (b + 3c) = e - (3b + 4c)$. From $a = d + 2(b - c)$ and $b - c = f - e$ we obtain $a = d + (b - c) + (f - e)$, which gives the relation $\mathbf{a} + \mathbf{c} + \mathbf{e} = \mathbf{b} + \mathbf{d} + \mathbf{f}$. □

### 4.2.2 Local formulas for discriminant

Here we define the discriminant of a cell in the dilinear topograph of a BQD $Q$ and show it is equal to $\Delta(Q)$.

**Definition 69.** If $\sigma = 2$ then

$$\Delta\left(\frac{a'}{d'}\Big)\frac{b}{c}\Big(\frac{a}{d}\right) = (2b-c)^2 - aa'.$$

Note that we get the quantity $(2c-b)^2 - dd'$ when we swap the top and bottom values of the cell. Moreover, both $(2b-c)^2 - aa'$ and $(2c-b)^2 - dd'$ remain unchanged after swapping the left and right pinwheel (i.e. after swapping $a, a'$ and $d, d'$).

Using the relations between the values in a cell (given by the arithmetic progressions and $a + c = b + d$) we can simplify the discriminant of a cell. For example, we can plug $a' = 2(2b+c) - a$ into $(2b-c)^2 - aa'$ and get a quantity involving only $a, b$ and $c$ (values in the right pinwheel).

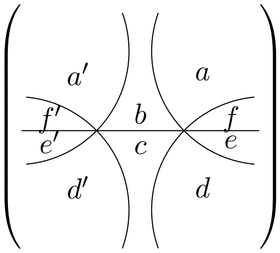
$$\begin{aligned}\Delta\left(\frac{a'}{d'}\Big)\frac{b}{c}\Big(\frac{a}{d}\right) = \Delta\left(\frac{b}{c}\Big(\frac{a}{d}\right) &= a^2 + (2b)^2 + c^2 - 2(2ab + ac + 2bc)\\ &= b^2 + (2c)^2 + d^2 - 2(2bc + bd + 2cd)\\ &= c^2 + (2d)^2 + a^2 - 2(2cd + ca + 2da)\\ &= d^2 + (2a)^2 + b^2 - 2(2da + db + 2ab).\end{aligned}$$

Note that by symmetry in swapping $a, a'$ and $d, d'$ we could also express it as a quantity involving only values in the left pinwheel. Moreover, adding the above quantities we can get a quantity involving **all** the values around a vertex. Note that this remains unchanged when we rotate or reflect the pinwheel. More precisely, we have

$$\Delta\left(\frac{a'}{d'}\Big)\frac{b}{c}\right) = \Delta\left(\frac{a'}{d'}\Big)\frac{b}{c}\Big(\frac{a}{d}\right) = \Delta\left(\frac{b}{c}\Big(\frac{a}{d}\right) \tag{4.1}$$

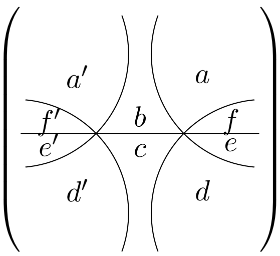
$$= \frac{3}{2}(a^2 + b^2 + c^2 + d^2) - 2(a+c)(b+d) - (ac + bd).$$

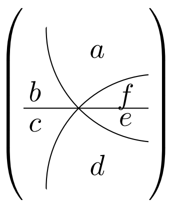
**Definition 70.** If $\sigma = 3$ then

$\Delta$ 


$$= (3b-c)^2 - aa' = \frac{(4b-3c)^2 - ff'}{4}.$$

As in the $\sigma = 2$ case we can simplify the discriminant of a cell and obtain the following.

$\Delta$ 


$= \Delta$ 


$$= a^2 + (3b)^2 + c^2 - 2(3ab + ac + 3bc)$$

$$= b^2 + (3c)^2 + d^2 - 2(3bc + bd + 3cd)$$

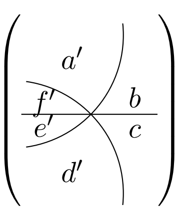
$$= c^2 + (3d)^2 + e^2 - 2(3cd + ce + 3de)$$

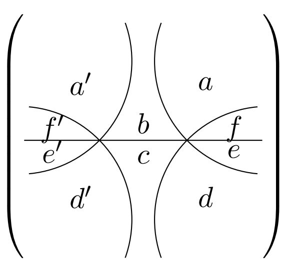
$$= d^2 + (3e)^2 + f^2 - 2(3de + df + 3ef)$$

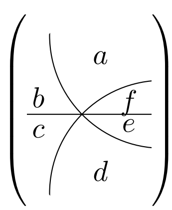
$$= e^2 + (3f)^2 + a^2 - 2(3ef + ea + 3fa)$$

$$= f^2 + (3a)^2 + b^2 - 2(3fa + fb + 3ab).$$

By adding the above quantities we obtain an expression for the discriminant of a cell which remains unchanged when we rotate or reflect a pinwheel contained in the cell.

$\Delta$ 


$= \Delta$ 


$= \Delta$ 


(4.2)

$$= \frac{11}{6}(a^2+b^2+c^2+d^2+e^2+f^2)-2(ab+bc+cd+de+ef+fa)-\frac{1}{3}(ac+ce+ea+bd+df+fb).$$

**Theorem 71.** *Let $Q(x,y) = \alpha x^2 + \beta\sqrt{\sigma}xy + \gamma y^2$ be a binary quadratic diform. The discriminants of all cells in the topograph of $Q$ are equal to $\Delta(Q) = \sigma(\beta^2\sigma - 4\alpha\gamma)$.*

*Proof.* It is easy to check using the above formulas that the discriminant of the home pinwheel equals $\Delta(Q)$. For instance, the discriminant at the home pinwheel in the case $\sigma = 2$ is given by

$$\Delta\left(\frac{\alpha}{\gamma}\left(\frac{2\alpha+2\beta+\gamma}{\alpha+2\beta+2\gamma}\right)\right) = 4\beta^2 - 8\alpha\gamma = \Delta(Q).$$

Then, by 4.1 and 4.2, the discriminants of all cells adjacent to the home pinwheel must also be equal to $\Delta(Q)$. Since all the other cells are linked to the home pinwheel, the discriminants of every cell must be equal to $\Delta(Q)$. □

### 4.2.3 Wells for definite diforms

The climbing principle says that when the values corresponding to an edge in the topograph are positive, then the arrows maintain a flow of constant increase. We show here that the topograph of a positive-definite BQD (whose values are always positive) has a unique source for its flow.

**Proposition 72.** *Let $Q$ be a positive-definite BQD over $R_\sigma$, with $\sigma = 2$ or $\sigma = 3$. Then the topograph of $Q$ exhibits a unique well – either a single vertex or an edge (double-well) from which all arrows emanate.*

*Proof.* The set of values occurring in the topograph of $Q$ is a set of positive integers, so it has a smallest element. Let $b$ be that smallest integer. Note that there might be two or more regions in the topograph of $Q$ where the same smallest value occurs.

Let $c$ be the smallest among the integers in a face opposite to $b$. The arithmetic progression rules, in a series of cases, imply that all arrow point away from the edge separating $b$ and $c$. When the edge separating $b$ and $c$ is marked by a circle, the topograph exhibits a double-well. Otherwise, the topograph exhibits a single-well. □

*Remark* 73. A double-well can be seen in Figure 4.2 above. A single-well can be seen in Figure 4.3 below.

### 4.2.4 Interlacing Conway topographs

Every value on the topograph of a BQD $Q$ appears on the topograph of $Q_{\text{red}}$ or of $Q_{\text{blue}}$. In this way, values from two of Conway's topographs interlace in the topograph of a binary quadratic diform.

**Proposition 74.** *If $z$ appears on the topograph of $Q_{\text{red}}$, then (1) $z$ appears on the topograph of $Q$ or (2) $\sigma \mid z$ and $z\sigma^{-1}$ appears on the topograph of both $Q_{\text{blue}}$ and $Q$. Similarly, if $z$ appears on the topograph of $Q_{\text{blue}}$, then (1) $z$ appears on the topograph of $Q$, or (2) $\sigma \mid z$ and $z\sigma^{-1}$ appears on the topographs of both $Q_{\text{red}}$ and $Q$.*

*Proof.* Suppose $z$ occurs on the topograph of $Q_{\text{red}}$. Thus $Q_{\text{red}}(u, v) = z$ for some coprime $u, v \in \mathbb{Z}$. If $\text{GCD}(u, \sigma v) = 1$, then $(u, v\sqrt{\sigma})$ is a primitive divector, and $Q(u, v\sqrt{\sigma}) = Q_{\text{red}}(u, v) = z$ appears in the topograph of $Q$.

If $\text{GCD}(u, \sigma v) \neq 1$, then $\sigma \mid u$ and $\text{GCD}(\sigma^{-1}u, v) = 1$. We compute $\sigma^{-1}z =$

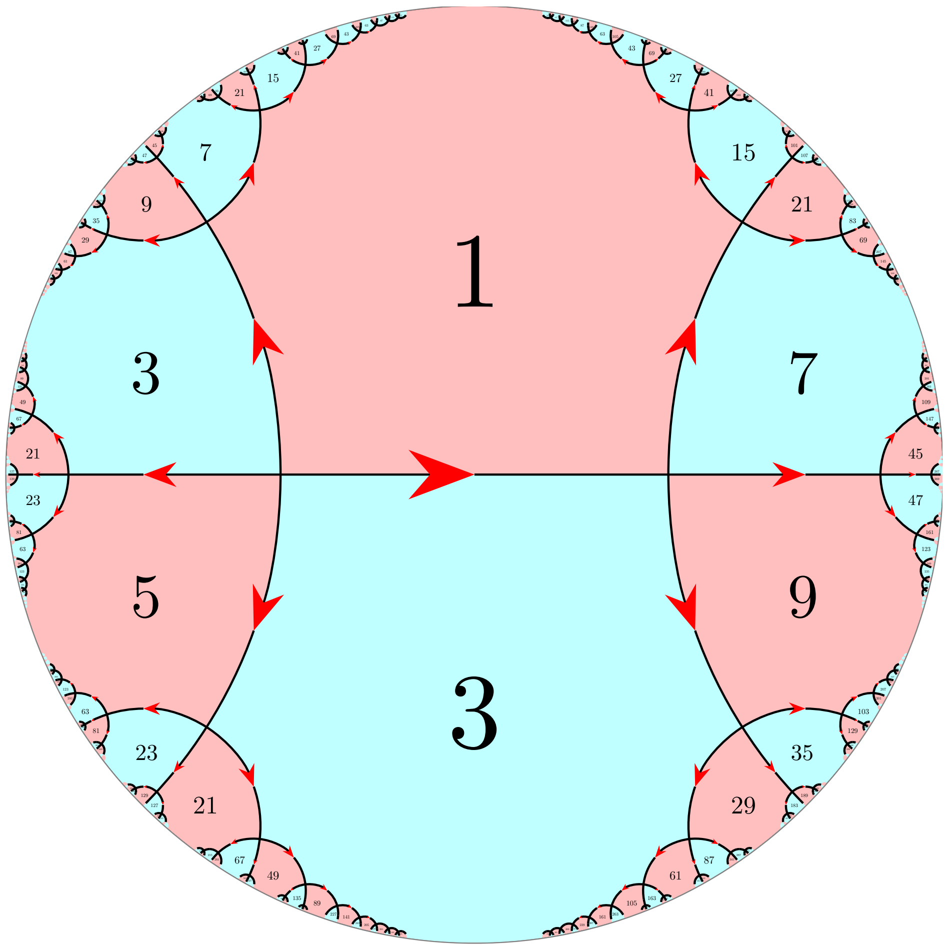



Figure 4.3: Topograph for the definite binary quadratic diform $Q(x, y) = x^2 + \sqrt{2}xy + 3y^2$ over $\mathbb{Z}[\sqrt{2}]$.

$\sigma^{-1}Q_{\text{red}}(u,v) = Q_{\text{blue}}(\sigma^{-1}u, v) = Q(\sigma^{-1}u\sqrt{\sigma}, v)$. Hence $\sigma^{-1}z$ appears on the topograph of both $Q_{\text{blue}}$ and $Q$. □

**Corollary 75.** *Let $\mu_{\text{red}}$ and $\mu_{\text{blue}}$ be the minimum nonzero absolute values of $Q_{\text{red}}$ and $Q_{\text{blue}}$. Then $\min\{\mu_{\text{red}}, \mu_{\text{blue}}\}$ is the minimum nonzero absolute value of $Q$.*

*Proof.* Every value on the topograph of $Q$ occurs in the topograph of $Q_{\text{red}}$ or $Q_{\text{blue}}$. Hence the minimum nonzero absolute value $\mu_Q$ of $Q$ satisfies $\mu_Q \geq \min\{\mu_{\text{red}}, \mu_{\text{blue}}\}$. Conversely, suppose without loss of generality that $\mu_{\text{red}} \leq \mu_{\text{blue}}$. Then either $\mu_{\text{red}}$ occurs in the topograph of $Q$, or else $\mu_{\text{red}}\sigma^{-1}$ occurs in the topograph of $Q_{\text{blue}}$. The latter would contradict the assumption that $\mu_{\text{red}} \leq \mu_{\text{blue}}$; thus $\mu_{\text{red}} = \mu_Q$. □

# Chapter 5

# Indefinite diforms

## 5.1 Indefinite forms and the river

If $Q(x,y) = \alpha x^2 + \beta\sqrt{\sigma}xy + \gamma y^2$ is a diform of discriminant $\Delta(Q)$ then the following identity holds:

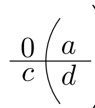
$$4\alpha\sigma Q(x,y) = (2\alpha\sqrt{\sigma}x + \beta\sigma y)^2 - \Delta(Q)y^2.$$

If $\Delta(Q) > 0$ then $Q$ represents both positive and negative integers, and it is called *indefinite* diform. We call $Q$ *degenerate* if $\Delta(Q) = 0$ or $\Delta(Q)$ is a square.

The faces of the topograph with value zero are called *lakes*. The set of segments separating positive values from negative values is called a *river*. Throughout this chapter we will explore the topographs of indefinite forms. We will show in this section that for indefinite forms, topographs without lakes have endless rivers.

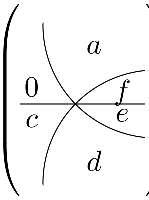
**Lemma 76.** *Let $Q$ be a BQD of discriminant $\Delta(Q)$. The topograph of $Q$ contains a lake if and only if $\Delta(Q)$ is a square.*

*Proof.* We compute the discriminant at a pinwheel containing a face with value zero using the formulas we got in the previous chapter.

If $\sigma = 2$ then $\Delta(Q) = \Delta$ (  )

$$= a^2 + (2 \cdot 0)^2 + c^2 - 2(2a \cdot 0 + ac + 2 \cdot 0 \cdot c)$$

$$= a^2 + c^2 - 2ac = (a-c)^2.$$

If $\sigma = 3$ then $\Delta(Q) = \Delta$ (  )

$$= a^2 + (3 \cdot 0)^2 + c^2 - 2(3a \cdot 0 + ac + 3 \cdot 0 \cdot c)$$

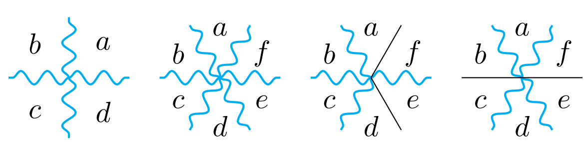
$$= a^2 + c^2 - 2ac = (a-c)^2.$$

For the converse, assume $\Delta(Q)$ is a square. Then the binary quadratic forms $Q_{\text{red}}$ and $Q_{\text{blue}}$ have square discriminants, so their topographs contain lakes; see ref. [11], Proposition 11.2. Hence the topograph of $Q$ contains a lake.

□

**Lemma 77.** *Rivers cannot branch.*

*Proof.* If a river branched, the faces around the branch point would alternate signs as they cross each river segment. Hence the rivers may only branch with even degree at a vertex. The possibilities, up to symmetry, are displayed below.



(I) (II) (III) (IV)

We use Corollary 68 to exclude each of them. When $\sigma = 2$, the identity $a + c = b + d$ yields a contradiction if the signs of $a$ and $c$ are equal, and opposite to the signs of $b$ and $d$. Similarly, when $\sigma = 3$, the identity $a + c + e = b + d + f$ yields a contradiction if the signs of $a, c, e$ are equal and opposite to the signs of $b, d, f$. Branch-forms (I) and (II) are excluded.

When $\sigma = 3$, then $a+d = b+e = c+f$ also holds. Thus we find a contradiction if the signs of $b, e$ are equal and opposite to the signs of $c, f$. This excludes form (III). We also find a contradiction if the signs of $a, d$ are equal, and opposite to the signs of $b, e$. This excludes form (IV). Hence the river cannot branch. □

**Proposition 78.** *If $Q$ is a nondegenerate indefinite diform, then its topograph contains a single endless nonbranching river.*

*Proof.* Since $Q$ is an indefinite diform both positive and negative values occur in its topograph. Since $Q$ is nondegenerate its discriminant is nonsquare and Lemma 76 implies the topograph of $Q$ does not contain a lake (i.e. zero does not occur).

As one travels from a positive face to a negative face, one must at some point cross a river from positive to negative. This gives existence. The climbing principle (propagation of growth-arrows) demonstrates that as one travels away from a river, one cannot hit another river, giving uniqueness.

The river cannot terminate since there are no lakes, and so the river is endless. The crux of the proposition is that rivers cannot branch. This is Lemma 77, proved above. □

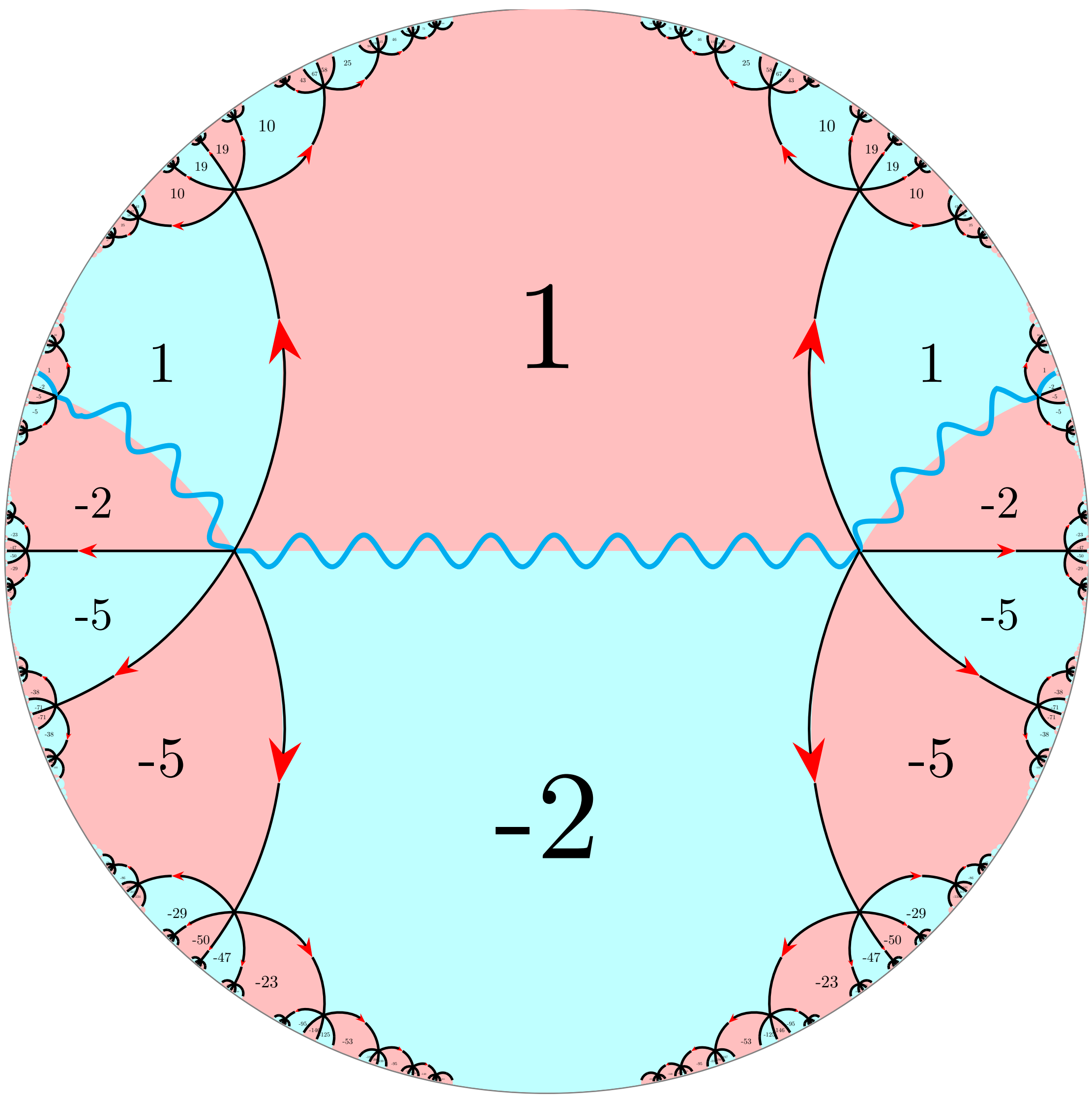



Figure 5.1: Topograph for the indefinite diform $Q(x, y) = x^2 - 2y^2$ over $\mathbb{Z}[\sqrt{3}]$.

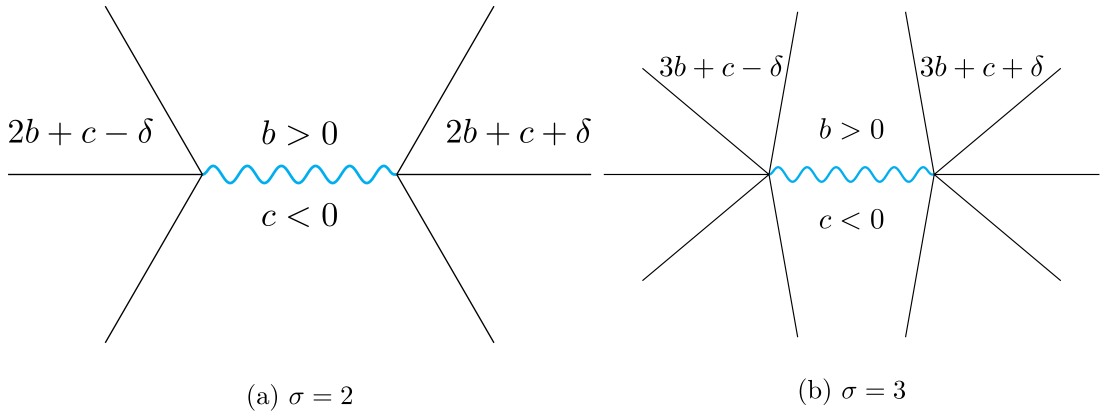



Figure 5.2: Cells containing a river segment.

The following result shows that going along the river, from segment to segment, we must find a repetition. Once we find a repetition, the pattern repeats.

**Proposition 79.** *Endless rivers are periodic.*

*Proof.* We will show that the values adjacent to the river in a river-cell of discriminant $\Delta$ are bounded by $k\Delta$ for some $k \in \mathbb{R}$. The river-cells are displayed in Figure 5.2 above. Let $\sigma = 2, 3$ and let $\delta$ be the step size of the arithmetic progressions as in Theorem 66. Then $\Delta = (\sigma b - c)^2 - (\sigma b + c - \delta)(\sigma b + c + \delta) = \delta^2 - 4\sigma bc$. Since $b > 0$ and $c < 0$ we have $-4\sigma bc > 0$ and thus $0 \leq \delta^2 < \Delta$. Moreover,

$$0 < b \cdot |c| \leq \frac{\Delta}{4\sigma}.$$

Thus $\delta < \sqrt{\Delta}$ and $b \leq \frac{\Delta}{4\sigma}$. Given $\Delta$, the value of $\delta$ and $b$ determine the value of $c$. Thus there are finitely many possible river-cells, so there must be a repetition. Hence the river is periodic. $\square$

## 5.2 Riverbends shapes and estimates

By Proposition 49 every region of the topograph is incident with infinitely many edges and the values around it form a bi-infinite sequence whose $n^{\text{th}}$ term is given by $a_n = Q(\vec{w} + n\sqrt{\sigma}\vec{v})$ for $n \in \mathbb{Z}$; see Figure 5.3 below.

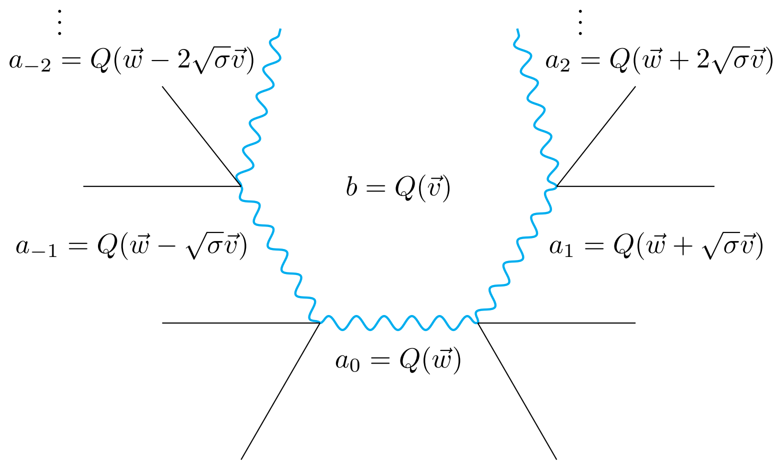



Figure 5.3: River around the edges of an infinity-gon.

**Proposition 80.** *The sequence $(a_n)_{n\in\mathbb{Z}}$ of values around a face labelled $b$, as in Figure 5.3, is a quadratic sequence with acceleration $2\sigma b$.*

*Proof.* We need to check that the sequence of differences between any two consecutive terms form an arithmetic progression of step size $2\sigma b$. The Arithmetic Progression Rule implies that the triples

$$(a_{-2}, \sigma b + a_{-1}, a_0), (a_{-1}, \sigma b + a_0, a_1) \text{ and } (a_0, \sigma b + a_1, a_2)$$

are arithmetic progressions.

Then $(\sigma b + a_{-1}) - a_{-2} = a_0 - (\sigma b + a_{-1})$, $(\sigma b + a_0) - a_{-1} = a_1 - (\sigma b + a_0)$, and $(\sigma b + a_1) - a_0 = a_2 - (\sigma b + a_1)$. This implies

$$(a_0 - a_{-1}) - (a_{-1} - a_{-2}) = 2\sigma b,$$

$$(a_1 - a_0) - (a_0 - a_{-1}) = 2\sigma b,$$

$$(a_2 - a_1) - (a_1 - a_0) = 2\sigma b.$$

Therefore $(a_2 - a_1), (a_1 - a_0), (a_0 - a_{-1}), (a_{-1} - a_{-2})$ is an arithmetic progression of step size $2\sigma b$. Hence $..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...$ is a quadratic sequence with acceleration $2\sigma b$. □

If the value $b$ is positive, then the values across the river are negative. But in this case, the values $a_{-2}, a_{-1}, a_0, a_1, a_2$ form a quadratic progression with positive acceleration $2\sigma b$. Similarly, if the value $b$ is negative, the values $a_{-2}, a_{-1}, a_0, a_1, a_2$ form a quadratic progression with negative acceleration $2\sigma b$. Hence as one travels far enough, to the left and to the right, we must see a sign switch for the values across from $b$. Since the entire river cannot be adjacent to a single region, the river must "bend."

As we have an endless nonbranching river, analysis of "riverbends" gives a minimum value bound for diforms.

**Theorem 81.** *Let $Q$ be a nondegenerate indefinite BQD, and let $\mu_Q$ denote its minimum nonzero absolute value.*

$\sigma = 2$*: If $Q$ is not $DL_2(R_\sigma)$-equivalent to a multiple of $x^2 - y^2$, then $\mu_Q \leq \sqrt{\Delta/10}$.*

$\sigma = 3$*: If $Q$ is not $DL_2(R_\sigma)$-equivalent to a multiple of $x^2 - y^2$, then $\mu_Q \leq \sqrt{2\Delta/25}$.*

*Proof.* If one finds riverbends as in Figures 5.4 or 5.5, the local formulas for discriminant give the stated minimum value bound or better. For example, the bound displayed in Figure 5.4 is obtained by expanding the discriminant and writing it is a sum of ten positive integers:

$$\Delta = b^2 + b^2 + b^2 + b^2 - bc - bc - bc - bc + c^2 - aa'.$$

Thus $|b| \cdot |b| \leq \Delta/10$, $|b| \cdot |c| \leq \Delta/10$, $|c| \cdot |c| \leq \Delta/10$ or $|a| \cdot |a'| \leq \Delta/10$. If the product of two positive integers is bounded by $\Delta/10$ then one of the two integers must be no greater than $\sqrt{\Delta/10}$. Hence the minimum nonzero absolute value of $Q$ satisfies $\mu_Q \leq \sqrt{\Delta/10}$. The bounds displayed in Figure 5.5 can be proved similarly.

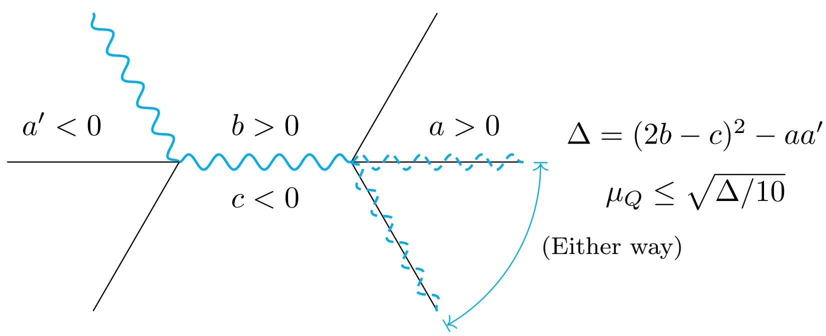



Figure 5.4: Riverbend types for $\sigma = 2$.

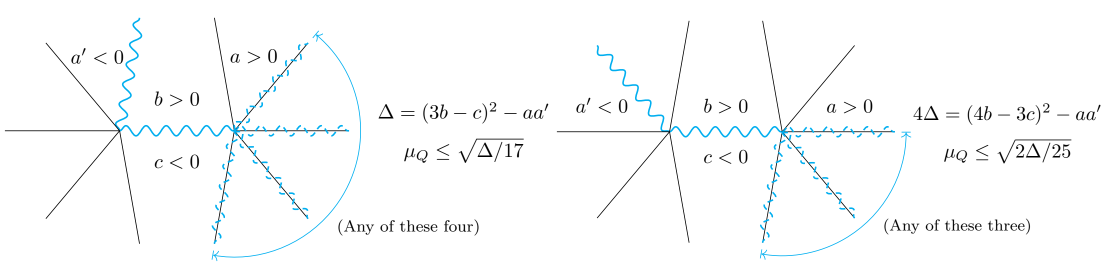



Figure 5.5: Riverbend types for $\sigma = 3$.

If no such riverbends of those shapes occur, then the river must maintain one of the three shapes of Figure 5.6 throughout its entire length.

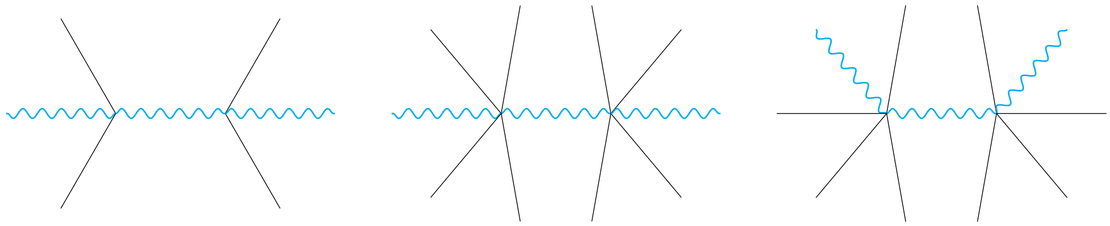


Figure 5.6: One more river shape for $\sigma = 2$ and two more shapes for $\sigma = 3$.

The isometry group of such a homogeneous river includes a translation along the river. Replacing $Q$ by a $DL_2(R_\sigma)$-equivalent form if necessary, we may place this river through the segment separating $\pm(1,0)$ and $\pm(0,1)$. Translation along the homogeneous rivers is then given by the matrices

$$R = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}, S = \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}, T = \begin{pmatrix} \sqrt{3} & 1 \\ 2 & \sqrt{3} \end{pmatrix}$$

in the three pictured cases of Figure 5.6. Periodicity of the river implies that $R^e$, $S^e$ or $T^e$ is an isometry of $Q$ for some $e > 0$.

The eigenvectors of $R$ and $S$ are $(1,1)$ and $(1,-1)$. If $\lambda$ and $\mu$ denote their eigenvalues, then

$$Q(1,1) = \lambda^{2e} Q(1,1) \text{ and } Q(1,-1) = \mu^{2e} Q(1,-1).$$

But a quick computation demonstrates that $\lambda, \mu \in \mathbb{R}$ and $\lambda, \mu \notin \{1, -1\}$. Hence $Q(1,1) = Q(1,-1) = 0$ in the two straight-river cases. Writing the diform as $Q(x,y) =$

$\alpha x^2 + \beta\sqrt{\sigma}xy + \gamma y^2$, this implies

$$\alpha + \beta\sqrt{\sigma} + \gamma = \alpha - \beta\sqrt{\sigma} + \gamma = 0.$$

Hence $\alpha = -\gamma$ and $\beta = 0$. We have proven that an endless straight river occurs only if $Q$ is equivalent to a multiple of $x^2 - y^2$ (when $\sigma = 2$ or $\sigma = 3$).

It remains to study the third shape of homogeneous river, on which $T$ acts by translation. The eigenvectors of $T$ are $(1, \sqrt{2})$ and $(-1, \sqrt{2})$, with eigenvalues $\sqrt{3} \pm \sqrt{2}$. Hence, if $T^e$ is an isometry of $Q$ for some $e > 0$, then $Q(1, \sqrt{2}) = Q(-1, \sqrt{2}) = 0$. In this case, $\alpha + \beta\sqrt{6} + 2\gamma = \alpha - \beta\sqrt{6} + 2\gamma = 0$. Hence $\beta = 0$ and $\alpha = -2\gamma$. We have proven that an endless homogeneous river of the third form occurs if and only if $Q$ is equivalent to a multiple of $2x^2 - y^2$. The discriminant of the diform $2x^2 - y^2$ is 24, while its minimum absolute value is $\mu_Q = 1$. The estimate $\mu_Q \leq \sqrt{2\Delta/25}$ can be directly checked in this case, finishing the proof. □

*Remark* 82. The exceptional diforms $x^2 - y^2$ cannot be removed from the previous theorem. The discriminant of the *diform* $x^2 - y^2$ is $4\sigma$ and its minimal value is $\mu_Q = 1$. Thus when $\sigma = 2$, the estimate $\mu_Q \leq \sqrt{\Delta/10}$ is violated; when $\sigma = 3$, the estimate $\mu_Q \leq \sqrt{2\Delta/25}$ is violated.

**Definition 83.** The *Markoff spectrum* is the set of real numbers $m = \mu_Q/\sqrt{\Delta}$ corresponding to all nondegenerate indefinite binary quadratic forms $Q$.

It has been long known that there is a gap in the Markoff spectrum between $1/\sqrt{12}$ and $1/\sqrt{13}$; see [3, §1, Proof of Theorem 3.3]. The following corollary follows directly from the previous theorem ($\sqrt{2/25}$ was replaced by $1/\sqrt{13}$) and Theorem 64.

**Corollary 84.** *Suppose that $Q_1$ and $Q_2$ are nondegenerate indefinite BQFs of discriminant $\Delta$, with $\sigma \mid \Delta$ and $[Q_2] = [A_\Delta] \cdot [Q_1]$. Then*

$\sigma = 2$*: If $Q_1$ and $Q_2$ are not equivalent to a multiple of $x^2 - 2y^2$, then*

$$\min\{\mu_{Q_1}, \mu_{Q_2}\} \leq \sqrt{\Delta/10}.$$

$\sigma = 3$*: If $Q_1$ and $Q_2$ are not equivalent to a multiple of $x^2 - 3y^2$, then*

$$\min\{\mu_{Q_1}, \mu_{Q_2}\} \leq \sqrt{\Delta/13}.$$

*Remark* 85. The classical bound for a binary quadratic form $Q$ is $\mu_Q \leq \sqrt{\Delta/5}$.

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